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TWO RATIONAL NODAL QUARTIC THREEFOLDS

IVAN CHELTSOV AND CONSTANTIN SHRAMOV

ABSTRACT. We prove that the quartic threefolds defined by

$$\sum_{i=0}^5 x_i = \sum_{i=0}^5 x_i^4 - t \left(\sum_{i=0}^5 x_i^2 \right)^2 = 0$$

in \mathbb{P}^5 are rational for $t = \frac{1}{6}$ and $t = \frac{7}{10}$.

1. INTRODUCTION

Consider the six-dimensional permutation representation \mathbb{W} of the group \mathfrak{S}_6 . Choose coordinates x_0, \dots, x_5 in \mathbb{W} so that they are permuted by \mathfrak{S}_6 . Then x_0, \dots, x_5 also serve as homogeneous coordinates in the projective space $\mathbb{P}^5 = \mathbb{P}(\mathbb{W})$.

Let us identify \mathbb{P}^4 with a hyperplane

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 0$$

in \mathbb{P}^5 . Denote by X_t the quartic threefold in \mathbb{P}^4 that is given by the equation

$$(1.1) \quad x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 = t \left(x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \right)^2,$$

where t is an element of the ground field, which we will always assume to be the field \mathbb{C} of complex numbers. Then X_t is singular for every $t \in \mathbb{C}$. Indeed, denote by Σ_{30} the \mathfrak{S}_6 -orbit of the point $[1 : 1 : \omega : \omega : \omega^2 : \omega^2]$, where $\omega = e^{\frac{2\pi i}{3}}$. Then $|\Sigma_{30}| = 30$, and X_t is singular at every point of Σ_{30} for every $t \in \mathbb{C}$ (see, for example, [12, Theorem 4.1]).

The possible singularities of the quartic threefold X_t have been described by van der Geer in [12, Theorem 4.1]. To recall his description, denote by \mathcal{L}_{15} the \mathfrak{S}_6 -orbit of the line that passes through the points $[1 : 0 : -1 : 1 : 0 : -1]$ and $[0 : 1 : -1 : 0 : 1 : -1]$, and denote by Σ_6 , Σ_{10} , and Σ_{15} the \mathfrak{S}_6 -orbits of the points $[-5 : 1 : 1 : 1 : 1 : 1]$, $[-1 : -1 : -1 : 1 : 1 : 1]$, and $[1 : -1 : 0 : 0 : 0 : 0]$, respectively. Then the curve \mathcal{L}_{15} is a union of fifteen lines, while $|\Sigma_6| = 6$, $|\Sigma_{10}| = 10$, and $|\Sigma_{15}| = 15$. Moreover, one has

$$\text{Sing}(X_t) = \begin{cases} \mathcal{L}_{15} & \text{if } t = \frac{1}{4}, \\ \Sigma_{30} \cup \Sigma_{15} & \text{if } t = \frac{1}{2}, \\ \Sigma_{30} \cup \Sigma_{10} & \text{if } t = \frac{1}{6}, \\ \Sigma_{30} \cup \Sigma_6 & \text{if } t = \frac{7}{10}, \\ \Sigma_{30} & \text{otherwise.} \end{cases}$$

Furthermore, if $t \neq \frac{1}{4}$, then all singular points of the quartic threefold X_t are isolated ordinary double points (nodes).

The threefold $X_{\frac{1}{2}}$ is classical. It is the so-called *Burkhardt quartic*. In [3], Burkhardt discovered that the subset $\Sigma_{30} \cup \Sigma_{15}$ is invariant under the action of the simple group $\mathrm{PSp}_4(\mathbf{F}_3)$ of order 25920. In [7], Coble proved that $\Sigma_{30} \cup \Sigma_{15}$ is the singular locus of the threefold $X_{\frac{1}{2}}$, and proved that $X_{\frac{1}{2}}$ is also $\mathrm{PSp}_4(\mathbf{F}_3)$ -invariant. Later Todd proved in [22] that $X_{\frac{1}{2}}$ is rational. In [15], de Jong, Shepherd-Barron, and Van de Ven proved that $X_{\frac{1}{2}}$ is the unique quartic threefold in \mathbb{P}^4 with 45 singular points.

The quartic threefold $X_{\frac{1}{4}}$ is also classical. It is known as the *Igusa quartic* from its modular interpretation as the Satake compactification of the moduli space of Abelian surfaces with level 2 structure (see [12]). The projectively dual variety of the quartic threefold $X_{\frac{1}{4}}$ is the so-called *Segre cubic*. Since the Segre cubic is rational, $X_{\frac{1}{4}}$ is rational as well.

During *Kullfest* conference dedicated to the 60th anniversary of Viktor Kulikov that was held in Moscow in December 2012, Alexei Bondal and Yuri Prokhorov posed

Problem 1.2. *Determine all $t \in \mathbb{C}$ such that X_t is rational.*

Since X_t is singular, we cannot apply Iskovskikh and Manin's theorem from [14] to X_t . Similarly, we cannot apply Mella's [18, Theorem 2] to X_t either, because the quartic threefold X_t is not \mathbb{Q} -factorial by [1, Lemma 2]. Nevertheless, Beauville proved

Theorem 1.3 ([1]). *If $t \notin \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{7}{10}\}$, then X_t is non-rational.*

Both $X_{\frac{1}{2}}$ and $X_{\frac{1}{4}}$ are rational. The goal of this paper is to prove

Theorem 1.4. *The quartic threefolds $X_{\frac{1}{6}}$ and $X_{\frac{7}{10}}$ are also rational.*

Surprisingly, the proof of Theorem 1.4 goes back to two classical papers of Todd. Namely, we will construct an explicit \mathfrak{A}_6 -birational map $\mathbb{P}^3 \dashrightarrow X_{\frac{7}{10}}$ that is a special case of Todd's construction from [20]. Similarly, we will construct an explicit \mathfrak{S}_5 -birational map $\mathbb{P}^3 \dashrightarrow X_{\frac{1}{6}}$ that is a degeneration of Todd's construction from [21]. We emphasize that our proof is self-contained, i.e. it does not rely on the results proved in [20] and [21], but recovers the necessary facts in our particular situation using additional symmetries arising from group actions.

Remark 1.5. Todd proved in [22] that the Burkhardt quartic $X_{\frac{1}{2}}$ is determinantal (see also [19, §5.1]). The constructions of our birational maps $\mathbb{P}^3 \dashrightarrow X_{\frac{7}{10}}$ and $\mathbb{P}^3 \dashrightarrow X_{\frac{1}{6}}$ imply that both $X_{\frac{7}{10}}$ and $X_{\frac{1}{6}}$ are determinantal (see [19, Example 6.4.2] and [19, Example 6.2.1]). Yuri Prokhorov pointed out that the quartic threefold

$$\det \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_4 & y_0 & y_3 & y_4 \\ y_2 & y_1 & y_1 & y_0 \\ y_0 & y_3 & y_2 & y_4 \end{pmatrix} = 0$$

in \mathbb{P}^4 with homogeneous coordinates y_0, \dots, y_4 has exactly 45 singular points. Thus, it is isomorphic to the Burkhardt quartic $X_{\frac{1}{2}}$ by [15]. It would be interesting to find similar determinantal equations of the threefolds $X_{\frac{7}{10}}$ and $X_{\frac{1}{6}}$.

The plan of the paper is as follows. In Section 2 we recall some preliminary results on representations of a central extension of the group \mathfrak{S}_6 , and some of its subgroups.

In Section 3 we collect results concerning a certain action of the group \mathfrak{A}_5 on \mathbb{P}^3 , and study \mathfrak{A}_5 -invariant quartic surfaces; the reason we pay so much attention to this group is that it is contained both in \mathfrak{A}_6 and in \mathfrak{S}_5 , and thus the information about its properties simplifies the study of the latter two groups. In Section 4 we collect auxiliary results about the groups \mathfrak{S}_6 , \mathfrak{A}_6 and \mathfrak{S}_5 , in particular about their actions on curves and their five-dimensional irreducible representations. In Section 5 we construct an \mathfrak{A}_6 -equivariant birational map $\mathbb{P}^3 \dashrightarrow X_{\frac{7}{10}}$. Finally, in Section 6 we construct an \mathfrak{S}_5 -equivariant birational map $\mathbb{P}^3 \dashrightarrow X_{\frac{1}{6}}$ and make some concluding remarks.

Throughout the paper, we denote a cyclic group of order n by μ_n , and we denote a dihedral group of order $2n$ by D_{2n} . In particular, one has $D_{12} \cong \mathfrak{S}_3 \times \mu_2$. By F_{36} we denote a group isomorphic to $(\mu_3 \times \mu_3) \rtimes \mu_4$, and by F_{20} we denote a group isomorphic to $\mu_5 \rtimes \mu_4$.

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2. REPRESENTATION THEORY

Recall that the permutation group \mathfrak{S}_6 has two central extensions $2^+\mathfrak{S}_6$ and $2^-\mathfrak{S}_6$ by the group μ_2 with the central subgroup contained in the commutator subgroup (see [8, p. xxiii] for details). We denote the first of them (i.e. the one where the preimages of a transposition in \mathfrak{S}_6 under the natural projection have order two) by $2.\mathfrak{S}_6$ to simplify notation. Similarly, for any group Γ we denote by $2.\Gamma$ a non-split central extension of Γ by the group μ_2 .

We start with recalling some facts about four- and five-dimensional representations of the group $2.\mathfrak{S}_6$ we will be working with. A reader who is not interested in details here can skip to Corollary 2.1, or even to Section 4 where we reformulate everything in geometric language. Also, we will see in Section 4 that our further constructions do not depend much on the choice of representations, and all computations one makes for one of them actually apply to all others.

Let \mathbb{I} and \mathbb{J} be the trivial and the non-trivial one-dimensional representations of the group \mathfrak{S}_6 , respectively. Consider the six-dimensional permutation representation \mathbb{W} of \mathfrak{S}_6 . One has

$$\mathbb{W} \cong \mathbb{I} \oplus \mathbb{W}_5 \otimes \mathbb{J}$$

for some irreducible representation \mathbb{W}_5 of \mathfrak{S}_6 . We can regard \mathbb{I} , \mathbb{J} and \mathbb{W}_5 as representations of the group $2.\mathfrak{S}_6$. Recall that there is a double cover $\mathrm{SL}_4(\mathbb{C}) \rightarrow \mathrm{SO}_6(\mathbb{C})$, see e.g. [10, Exercise 20.39]. Using it, we conclude that there is an embedding of the group $2.\mathfrak{S}_6$ into $\mathrm{SL}_4(\mathbb{C})$. This embedding gives rise to two four-dimensional representations of $2.\mathfrak{S}_6$ that differ by a tensor product with \mathbb{J} . We fix one of these two representations \mathbb{U}_4 . Note that

$$\mathbb{I} \oplus \mathbb{W}_5 \cong \Lambda^2(\mathbb{U}_4).$$

Recall that there are coordinates x_0, \dots, x_5 in \mathbb{W} that are permuted by the group \mathfrak{S}_6 . We will refer to a subgroup of $2.\mathfrak{S}_6$ fixing one of the corresponding points as a *standard* subgroup $2.\mathfrak{S}_5$; we denote any such subgroup by $2.\mathfrak{S}_5^{st}$. A subgroup of $2.\mathfrak{S}_6$ that is isomorphic to $2.\mathfrak{S}_5$ but is not conjugate to a standard $2.\mathfrak{S}_5$ will be called a *non-standard* subgroup $2.\mathfrak{S}_5$; we denote any such subgroup by $2.\mathfrak{S}_5^{nst}$. These agree with standard and non-standard subgroups of \mathfrak{S}_6 isomorphic to \mathfrak{S}_5 , although outer automorphisms of \mathfrak{S}_6 do not lift to $2.\mathfrak{S}_6$. Any subgroup of $2.\mathfrak{S}_6$ that is isomorphic to $2.\mathfrak{A}_5$, $2.\mathfrak{S}_4$ or $2.\mathfrak{A}_4$ and is contained in $2.\mathfrak{S}_5^{st}$ is denoted by $2.\mathfrak{A}_5^{st}$, $2.\mathfrak{S}_4^{st}$ or $2.\mathfrak{A}_4^{st}$, respectively. Similarly, any subgroup of $2.\mathfrak{S}_6$ that is isomorphic to $2.\mathfrak{A}_5$, $2.\mathfrak{S}_4$ or $2.\mathfrak{A}_4$ and is contained in $2.\mathfrak{S}_5^{nst}$ is denoted by $2.\mathfrak{A}_5^{nst}$, $2.\mathfrak{S}_4^{nst}$ or $2.\mathfrak{A}_4^{nst}$, respectively.

The values of characters of important representations of the group $2.\mathfrak{S}_6$, and the information about some of its subgroups are presented in Table 1, cf. [8, p. 5]. The first two columns of Table 1 describe conjugacy classes of elements of the group $2.\mathfrak{S}_6$. The first column lists the orders of the elements in the corresponding conjugacy class, and the second column, except for the entries in the second and the third row, gives a cycle type of the image of an element under projection to \mathfrak{S}_6 (for example, $[3, 2]$ denotes a product of two disjoint cycles of lengths 3 and 2). By id we denote the identity element of $2.\mathfrak{S}_6$, and z denotes the unique non-trivial central element of $2.\mathfrak{S}_6$. Note that the preimages of some of conjugacy classes in \mathfrak{S}_6 split into a union of two conjugacy classes in $2.\mathfrak{S}_6$. The next three columns list the values of the characters of the representations \mathbb{W} , \mathbb{W}_5 and \mathbb{U}_4 of $2.\mathfrak{S}_6$. Note that there is no real ambiguity in the choice of $\sqrt{-3}$ since we did not specify any way to distinguish the two conjugacy classes in $2.\mathfrak{S}_6$ whose elements are projected to cycles of length 6 in \mathfrak{S}_6 up to this point (note that the two ways to choose a sign here is exactly a tensor multiplication of the representation with \mathbb{J} , i.e. the choice between two homomorphisms of $2.\mathfrak{S}_6$ to $\text{SL}_4(\mathbb{C})$ having the same image). The remaining columns list the numbers of elements from each of the conjugacy classes of $2.\mathfrak{S}_6$ in subgroups of certain types. By $2.F_{36}$ (respectively, by $2.F_{20}$, by $2.D_{12}^{nst}$) we denote a subgroup of $2.\mathfrak{S}_6$ (respectively, of $2.\mathfrak{S}_6$, or of $2.\mathfrak{S}_5^{nst}$) isomorphic to a central extension of F_{36} (respectively, of F_{20} , or of D_{12}) by μ_2 . A subgroup $2.F_{20}$ is actually contained in a subgroup $2.\mathfrak{S}_5^{st}$ and in a subgroup $2.\mathfrak{S}_5^{nst}$. Note that the intersection of a conjugacy class in a group with a subgroup may (and often does) split into several conjugacy classes in this subgroup.

It is immediate to see from Table 1 that \mathbb{U}_4 is an irreducible representation of the group $2.\mathfrak{S}_6$. Using the information provided by Table 1, we immediately obtain the following results.

Corollary 2.1. *Let Γ be a subgroup of $2.\mathfrak{S}_6$. After restriction to the subgroup Γ the $2.\mathfrak{S}_6$ -representation \mathbb{U}_4*

- (i) *remains irreducible, if Γ is one of the subgroups $2.\mathfrak{A}_6$, $2.\mathfrak{S}_5^{nst}$, $2.\mathfrak{A}_5^{nst}$, $2.\mathfrak{S}_4^{nst}$, $2.F_{36}$, or $2.F_{20}$;*
- (ii) *splits into a sum of two non-isomorphic irreducible two-dimensional representations, if Γ is one of the subgroups $2.\mathfrak{A}_5^{st}$, $2.\mathfrak{A}_4^{nst}$, or $2.D_{12}^{nst}$.*

Proof. Compute inner products of the corresponding characters with themselves, and keep in mind that neither of the groups $2.\mathfrak{A}_5^{st}$, $2.\mathfrak{A}_4^{nst}$, and $2.D_{12}^{nst}$ has an irreducible three-dimensional representation with a non-trivial action of the central subgroup. \square

Remark 2.2. By Corollary 2.1(i), the $2.\mathfrak{S}_5^{nst}$ -representation \mathbb{U}_4 is irreducible. One can check that it is not induced from any proper subgroup of $2.\mathfrak{S}_5^{nst}$, i.e. it defines a primitive

TABLE 1. Characters and subgroups of the group $2.\mathfrak{S}_6$

ord	type	\mathbb{W}	\mathbb{W}_5	\mathbb{U}_4	$2.\mathfrak{S}_6$	$2.\mathfrak{A}_6$	$2.\mathfrak{S}_5^{nst}$	$2.\mathfrak{A}_5^{st}$	$2.\mathfrak{A}_5^{nst}$	$2.\mathfrak{S}_4^{nst}$	$2.\mathfrak{A}_4^{nst}$	$2.F_{36}$	$2.F_{20}$	$2.D_{12}^{nst}$
1	id	6	5	4	1	1	1	1	1	1	1	1	1	1
2	z	6	5	-4	1	1	1	1	1	1	1	1	1	1
2	$[2]$	4	-3	0	30	0	0	0	0	0	0	0	0	0
4	$[2, 2]$	2	1	0	90	90	30	30	30	6	6	18	10	6
4	$[2, 2, 2]$	0	1	0	30	0	20	0	0	12	0	0	0	8
6	$[3]$	3	2	2	40	40	0	20	0	0	0	4	0	0
3	$[3]$	3	2	-2	40	40	0	20	0	0	0	4	0	0
6	$[3, 2]$	1	0	0	120	0	0	0	0	0	0	0	0	0
6	$[3, 2]$	1	0	0	120	0	0	0	0	0	0	0	0	0
6	$[3, 3]$	0	-1	-1	40	40	20	0	20	8	8	4	0	2
3	$[3, 3]$	0	-1	1	40	40	20	0	20	8	8	4	0	2
8	$[4]$	2	-1	0	180	0	60	0	0	12	0	0	20	0
8	$[4, 2]$	0	-1	0	180	180	0	0	0	0	0	36	0	0
10	$[5]$	1	0	1	144	144	24	24	24	0	0	0	4	0
5	$[5]$	1	0	-1	144	144	24	24	24	0	0	0	4	0
12	$[6]$	0	1	$\sqrt{-3}$	120	0	20	0	0	0	0	0	0	2
12	$[6]$	0	1	$-\sqrt{-3}$	120	0	20	0	0	0	0	0	0	2

subgroup isomorphic to $2.\mathfrak{S}_5$ in $\mathrm{GL}_4(\mathbb{C})$. Note that this subgroup is not present in the list given in [9, §8.5]. It is still listed by some other classical surveys, see e.g. [2, §119].

Corollary 2.3. *Let Γ be a subgroup of $2.\mathfrak{S}_6$. After restriction to the subgroup Γ the $2.\mathfrak{S}_6$ -representation \mathbb{W}_5*

- (i) *remains irreducible, if Γ is one of the subgroups $2.\mathfrak{A}_6$, $2.\mathfrak{S}_5^{nst}$, or $2.\mathfrak{A}_5^{nst}$;*
- (ii) *splits into a sum of the trivial and an irreducible four-dimensional representation if Γ is a subgroup $2.\mathfrak{A}_5^{st}$;*
- (iii) *splits into a sum of the trivial and two different irreducible two-dimensional representations if Γ is a subgroup $2.\mathrm{D}_{12}^{nst}$.*

In the sequel we will denote the restrictions of the $2.\mathfrak{S}_6$ -representations \mathbb{U}_4 and \mathbb{W}_5 to various subgroups by the same symbols for simplicity. The next two corollaries are implied by direct computations (we used GAP software [11] to perform them).

Corollary 2.4. *The following assertions hold:*

- (i) *the \mathfrak{A}_6 -representation $\mathrm{Sym}^2(\mathbb{U}_4^\vee)$ does not contain one-dimensional subrepresentations;*
- (ii) *the \mathfrak{A}_6 -representation $\mathrm{Sym}^4(\mathbb{U}_4^\vee)$ does not contain one-dimensional subrepresentations;*
- (iii) *the \mathfrak{A}_5^{nst} -representation $\mathrm{Sym}^2(\mathbb{U}_4^\vee)$ splits into a sum of two different irreducible three-dimensional representations and one irreducible four-dimensional representation;*
- (iv) *the $2.\mathfrak{A}_5^{nst}$ -representation $\mathrm{Sym}^3(\mathbb{U}_4^\vee)$ does not contain one-dimensional subrepresentations;*
- (v) *the \mathfrak{A}_5^{nst} -representation $\mathrm{Sym}^4(\mathbb{U}_4^\vee)$ has a unique two-dimensional subrepresentation, and this subrepresentation splits into a sum of two trivial representations of \mathfrak{A}_5^{nst} .*

Recall that all representations of a symmetric group are self-dual. Therefore, to study invariant hypersurfaces in $\mathbb{P}(\mathbb{W}_5)$ we will use the following result.

Corollary 2.5. *Let Γ be one of the groups \mathfrak{S}_6 , \mathfrak{A}_6 or \mathfrak{S}_5^{nst} . Then*

- (i) *the Γ -representation $\mathrm{Sym}^2(\mathbb{W}_5)$ has a unique one-dimensional subrepresentation;*
- (ii) *the Γ -representation $\mathrm{Sym}^4(\mathbb{W}_5)$ has a unique two-dimensional subrepresentation, and this subrepresentation splits into a sum of two trivial representations of Γ .*

We conclude this section by recalling some information about several subgroups of $2.\mathfrak{S}_6$ that are smaller than those listed in Table 1. Namely, we list in Table 2 orders, types and numbers of elements in certain subgroups of $2.\mathfrak{A}_5^{nst}$. We keep the notation used in Table 1. By $2.\mathfrak{S}_3$ we denote a subgroup of $2.\mathfrak{A}_5^{nst}$ isomorphic to $2.\mathfrak{S}_3$. Note that the preimage in $2.\mathfrak{S}_6$ of any subgroup $\mu_5 \subset \mathfrak{S}_6$ is isomorphic to μ_{10} .

Looking at Table 2 (and keeping in mind character values provided by Table 1) we immediately obtain the following.

Corollary 2.6. *Let Γ be a subgroup of $2.\mathfrak{A}_5^{nst} \subset 2.\mathfrak{S}_6$. After restriction to Γ the $2.\mathfrak{S}_6$ -representation \mathbb{U}_4*

- (i) *splits into a sum of two non-isomorphic irreducible two-dimensional representations if Γ is a subgroup $2.\mathrm{D}_{10}$;*
- (ii) *splits into a sum of an irreducible two-dimensional representation and two non-isomorphic one-dimensional representations if Γ is a subgroup $2.\mathfrak{S}_3'$;*

TABLE 2. Subgroups of $2.\mathfrak{A}_5^{nst}$

ord	type	$2.D_{10}$	$2.\mathfrak{S}'_3$	$2.(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$	$\boldsymbol{\mu}_{10}$
1	id	1	1	1	1
2	z	1	1	1	1
4	$[2, 2]$	10	6	6	0
6	$[3, 3]$	0	2	0	0
3	$[3, 3]$	0	2	0	0
10	$[5]$	4	0	0	4
5	$[5]$	4	0	0	4

- (iii) *splits into a sum of two isomorphic irreducible two-dimensional representations if Γ is a subgroup $2.(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$;*
- (iv) *splits into a sum of four pairwise non-isomorphic one-dimensional representations if Γ is a subgroup $\boldsymbol{\mu}_{10}$.*

3. ICOSAHEDRAL GROUP IN THREE DIMENSIONS

In this section, we consider the action of the group \mathfrak{A}_5 on the projective space \mathbb{P}^3 arising from a non-standard embedding of $\mathfrak{A}_5 \hookrightarrow \mathfrak{S}_6$. Namely, we identify \mathbb{P}^3 with the projectivization $\mathbb{P}(\mathbb{U}_4)$, where \mathbb{U}_4 is the restriction of the four-dimensional irreducible representation of the group $2.\mathfrak{S}_6$ introduced in Section 2 to a subgroup $2.\mathfrak{A}_5^{nst}$ (which we will refer to as just $2.\mathfrak{A}_5$ in this section). Recall from Corollary 2.1(i) that \mathbb{U}_4 is an irreducible representation of $2.\mathfrak{A}_5$.

Remark 3.1 (see e. g. [8, p. 2]). Let Γ be a proper subgroup of \mathfrak{A}_5 such that the index of Γ is at most 15. Then Γ is isomorphic either to \mathfrak{A}_4 , or to D_{10} , or to \mathfrak{S}_3 , or to $\boldsymbol{\mu}_5$, or to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. In particular, if \mathfrak{A}_5 acts transitively on the set of $r < 15$ elements, then $r \in \{5, 6, 10, 12\}$.

Lemma 3.2. *Let Ω be an \mathfrak{A}_5 -orbit of length $r \leq 15$ in \mathbb{P}^3 . Then either $r = 10$, or $r = 12$. Moreover, \mathbb{P}^3 contains exactly two \mathfrak{A}_5 -orbits of length 10 and exactly two \mathfrak{A}_5 -orbits of length 12.*

Proof. By Remark 3.1 one has $r \in \{1, 5, 6, 10, 12, 15\}$. The case $r = 1$ is impossible since \mathbb{U}_4 is an irreducible $2.\mathfrak{A}_5$ -representation. Restricting \mathbb{U}_4 to subgroups of $2.\mathfrak{A}_5$ isomorphic to $2.\mathfrak{A}_4$, $2.D_{10}$, and $2.(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$, and applying Corollaries 2.1(ii) and 2.6(i), (iii), we see that $r \notin \{5, 6, 15\}$.

Restricting \mathbb{U}_4 to a subgroup of $2.\mathfrak{A}_5$ isomorphic to $2.\mathfrak{S}_3$, applying Corollary 2.6(ii) and keeping in mind that there are ten subgroups isomorphic to \mathfrak{S}_3 in \mathfrak{A}_5 , we see that \mathbb{P}^3 contains exactly two \mathfrak{A}_5 -orbits of length 10.

Finally, restricting \mathbb{U}_4 to a subgroup of $2.\mathfrak{A}_5$ isomorphic to $\boldsymbol{\mu}_{10}$, applying Corollary 2.6(iv) and keeping in mind that there are six subgroups isomorphic to $\boldsymbol{\mu}_5$ in \mathfrak{A}_5 , we see that \mathbb{P}^3 contains exactly two \mathfrak{A}_5 -orbits of length 12. \square

Lemma 3.3. *There are no \mathfrak{A}_5 -invariant surfaces of degree at most three in \mathbb{P}^3 .*

Proof. Apply Corollary 2.4(iii), (iv). \square

By Corollary 2.1(ii), the subgroup $\mathfrak{A}_4 \subset \mathfrak{A}_5$ leaves invariant two disjoint lines in \mathbb{P}^3 , say L_1 and L'_1 . Let L_1, \dots, L_5 be the \mathfrak{A}_5 -orbit of the line L_1 , and let L'_1, \dots, L'_5 be the \mathfrak{A}_5 -orbit of the line L'_1 .

Lemma 3.4. *The lines L_1, \dots, L_5 (respectively, the lines L'_1, \dots, L'_5) are pairwise disjoint.*

Proof. Suppose that some of the lines L_1, \dots, L_5 have a common point. Since the action of \mathfrak{A}_5 on the set $\{L_1, \dots, L_5\}$ is doubly transitive, this implies that every two of the lines L_1, \dots, L_5 have a common point. Therefore, either all lines L_1, \dots, L_5 are coplanar, or all of them pass through one point. Both of these cases are impossible since the $2\mathfrak{A}_5$ -representation \mathbb{U}_4 is irreducible by Corollary 2.1(i). Therefore, the lines L_1, \dots, L_5 are pairwise disjoint. A similar argument applies to the lines L'_1, \dots, L'_5 . \square

Corollary 3.5. *Any \mathfrak{A}_5 -orbit contained in the union $L_1 \cup \dots \cup L_5$ has length at least 20.*

Proof. Corollary 2.1(ii) implies that the stabilizer $\Gamma \cong \mathfrak{A}_4$ of the line L_1 acts on L_1 faithfully. Therefore, the length of any Γ -orbit contained in L_1 is at least four. Thus the required assertion follows from Lemma 3.4. \square

We are going to describe the configuration formed by the lines $L_1, \dots, L_5, L'_1, \dots, L'_5$.

Definition 3.6. Let $T_1, \dots, T_5, T'_1, \dots, T'_5$ be different lines in a projective space. We say that they form a *double five configuration* if the following conditions hold:

- the lines T_1, \dots, T_5 (respectively, the lines T'_1, \dots, T'_5) are pairwise disjoint;
- for every i the lines T_i and T'_i are disjoint;
- for every $i \neq j$ the line T_i meets the line T'_j .

Lemma 3.7. *The lines $L_1, \dots, L_5, L'_1, \dots, L'_5$ form a double five configuration. Moreover, the only line in \mathbb{P}^3 that intersects all lines of L_1, \dots, L_5 but L_i is the line L'_i , and the only line in \mathbb{P}^3 that intersects all lines of L'_1, \dots, L'_5 but L'_i is the line L_i .*

Proof. For any i the lines L_i and L'_i are disjoint by construction. The lines L_1, \dots, L_5 (respectively, the lines L'_1, \dots, L'_5) are pairwise disjoint by Lemma 3.4.

Since any three pairwise skew lines in \mathbb{P}^3 are contained in a smooth quadric surface, and an intersection of two different quadric surfaces in \mathbb{P}^3 cannot contain three pairwise skew lines, we see that for any three indices $1 \leq i < j < k \leq 5$ there is a unique quadric surface Q_{ijk} in \mathbb{P}^3 passing through the lines L_i, L_j and L_k . Moreover, the quadric Q_{ijk} is smooth. Note also that the quadric Q_{ijk} is not \mathfrak{A}_5 -invariant by Lemma 3.3. This implies that all five lines L_1, \dots, L_5 are not contained in a quadric.

Therefore, we may assume that the quadric Q_{123} does not contain the line L_4 . It is well-known that in this case either there is a unique line L meeting all four lines L_1, \dots, L_4 , or there are exactly two lines L and L' meeting L_1, \dots, L_4 . In the latter case the stabilizer $\Gamma \subset \mathfrak{A}_5$ of the quadruple L_1, \dots, L_4 (i.e. the stabilizer of the line L_5) preserves the lines L_5, L and L' . On the other hand, the lines L and L' are different from L_5 since L_5 meets neither of the lines L_1, \dots, L_4 ; moreover, the group $\Gamma \cong \mathfrak{A}_4$ fixes both L and L' . But Γ cannot fix three different lines in \mathbb{P}^3 by Corollary 2.1(ii). The contradiction shows that there is a unique line L meeting L_1, \dots, L_4 . Again we see that $L \neq L_5$, so that $L = L'_5$ by Corollary 2.1(ii).

Since the group \mathfrak{A}_5 permutes the lines L_1, \dots, L_5 transitively, we conclude that the only line in \mathbb{P}^3 that intersects all lines of L_1, \dots, L_5 except L_i is the line L'_i . Similarly, we see that the only line in \mathbb{P}^3 that intersects all lines of L'_1, \dots, L'_5 except L'_i is the line L_i . In particular, the lines $L_1, \dots, L_5, L'_1, \dots, L'_5$ form a double five configuration. \square

Lemma 3.8. *Every \mathfrak{A}_5 -invariant curve of degree at most three in \mathbb{P}^3 is a twisted cubic. Moreover, there are exactly two \mathfrak{A}_5 -invariant twisted cubic curves in \mathbb{P}^3 .*

Proof. Let C be an \mathfrak{A}_5 -invariant curve of degree at most three in \mathbb{P}^3 . Since the $2\mathfrak{A}_5$ -representation \mathbb{U}_4 is irreducible, we conclude that C is a twisted cubic.

By Corollary 2.4(iii), one has

$$(3.9) \quad \mathrm{Sym}^2(\mathbb{U}_4) \cong \mathbb{V}_3 \oplus \mathbb{V}'_3 \oplus \mathbb{V}_4,$$

where \mathbb{V}_3 , \mathbb{V}'_3 , and \mathbb{V}_4 , are irreducible representations of the group \mathfrak{A}_5 of dimensions 3, 3, and 4, respectively. Note that \mathbb{V}_3 and \mathbb{V}'_3 are not isomorphic.

Denote by \mathcal{Q} and \mathcal{Q}' the linear systems of quadrics in \mathbb{P}^3 that correspond to \mathbb{V}_3 and \mathbb{V}'_3 , respectively. Since \mathbb{P}^3 does not contain \mathfrak{A}_5 -orbits of lengths less or equal to eight by Lemma 3.2, we see that the base loci of \mathcal{Q} and \mathcal{Q}' contain \mathfrak{A}_5 -invariant curves \mathcal{C}^1 and \mathcal{C}^2 , respectively. The degrees of these curves must be less than four, so that they are twisted cubic curves. This also implies that the base loci of \mathcal{Q} and \mathcal{Q}' are exactly the curves \mathcal{C}^1 and \mathcal{C}^2 , respectively.

Now take an arbitrary \mathfrak{A}_5 -invariant twisted cubic curve C in \mathbb{P}^3 . The quadrics in \mathbb{P}^3 passing through C define a three-dimensional \mathfrak{A}_5 -subrepresentation in $\mathrm{Sym}^2(\mathbb{U}_4)$. Moreover, different \mathfrak{A}_5 -invariant twisted cubics give different \mathfrak{A}_5 -subrepresentations of $\mathrm{Sym}^2(\mathbb{U}_4)$. Thus, (3.9) implies that C coincides either with \mathcal{C}^1 or with \mathcal{C}^2 . \square

Keeping in mind Lemma 3.8, we will denote the two \mathfrak{A}_5 -invariant twisted cubic curves in \mathbb{P}^3 by \mathcal{C}^1 and \mathcal{C}^2 throughout this section.

Remark 3.10. The curves \mathcal{C}^1 and \mathcal{C}^2 are disjoint. Indeed, otherwise, their intersection would contain at least 12, which is impossible, since a twisted cubic curve is an intersection of quadrics.

The lines in \mathbb{P}^3 that are tangent to the curves \mathcal{C}^1 and \mathcal{C}^2 sweep out quartic surfaces \mathcal{S}^1 and \mathcal{S}^2 , respectively. These surfaces are \mathfrak{A}_5 -invariant. The singular loci of \mathcal{S}^1 and \mathcal{S}^2 are the curves \mathcal{C}^1 and \mathcal{C}^2 , respectively. In particular, the surfaces \mathcal{S}^1 and \mathcal{S}^2 are different. Their singularities along these curves are locally isomorphic to a product of \mathbb{A}^1 and an ordinary cusp.

Denote by \mathcal{P} the pencil of quartics in \mathbb{P}^3 generated by \mathcal{S}^1 and \mathcal{S}^2 .

Lemma 3.11. *All \mathfrak{A}_5 -invariant quartic surfaces in \mathbb{P}^3 are contained in the pencil \mathcal{P} .*

Proof. This follows from Corollary 2.4(v). \square

We proceed by describing the base locus of the pencil \mathcal{P} . This was done in [4, Remark 2.6], but we reproduce the details here for the convenience of the reader.

Lemma 3.12. *The base locus of the pencil \mathcal{P} is an irreducible curve B of degree 16. It has 24 singular points, these points are in a union of two \mathfrak{A}_5 -orbits of length 12, and each of them is an ordinary cusp of the curve B . The curve B contains a unique \mathfrak{A}_5 -orbit of length 20.*

Proof. Denote by B the base curve of the pencil \mathcal{P} . Let us show that the curves \mathcal{C}^1 and \mathcal{C}^2 are not contained in B . Since \mathcal{C}^1 is projectively normal, there is an exact sequence of $2\mathfrak{A}_5$ -representations

$$0 \rightarrow H^0(\mathcal{I}_{\mathcal{C}^1} \otimes \mathcal{O}_{\mathbb{P}^3}(4)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \rightarrow H^0(\mathcal{O}_{\mathcal{C}^1} \otimes \mathcal{O}_{\mathbb{P}^3}(4)) \rightarrow 0,$$

where $\mathcal{I}_{\mathcal{C}^1}$ is the ideal sheaf of \mathcal{C}^1 . The $2\mathfrak{A}_5$ -representation $H^0(\mathcal{O}_{\mathcal{C}^1} \otimes \mathcal{O}_{\mathbb{P}^3}(4))$ contains a one-dimensional subrepresentation corresponding to the unique \mathfrak{A}_5 -orbit of length 12 in $\mathcal{C}^1 \cong \mathbb{P}^1$. This shows that \mathcal{P} contains a surface that does not pass through \mathcal{C}^1 , so that \mathcal{C}^1 is not contained in B . Similarly, we see that \mathcal{C}^2 is not contained in B .

Let $\rho: \hat{\mathcal{S}}^1 \rightarrow \mathcal{S}^1$ be the normalization of the surface \mathcal{S}^1 , and let $\hat{\mathcal{C}}^1$ be the preimage of the curve \mathcal{C}^1 via ρ . Then the action of the group \mathfrak{S}_5 , and in particular of its subgroup \mathfrak{A}_5 , lifts to $\hat{\mathcal{S}}^1$. One has $\hat{\mathcal{S}}^1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $\rho^*(\mathcal{O}_{\mathcal{S}^1} \otimes \mathcal{O}_{\mathbb{P}^3}(1))$ is a divisor of bi-degree $(1, 2)$. This shows that $\hat{\mathcal{C}}^1$ is of bi-degree $(1, 1)$. Thus, the action of \mathfrak{A}_5 on $\hat{\mathcal{S}}$ is diagonal by [6, Lemma 6.4.3(i)].

Denote by \hat{B} be the preimage of the curve B via ρ . Then \hat{B} is a divisor of bi-degree $(4, 8)$. Hence, the curve \hat{B} is irreducible by [6, Lemma 6.4.4(i)], so that the curve B is irreducible as well.

Note that the curve \hat{B} is singular. Indeed, the intersection $\mathcal{S}^1 \cap \mathcal{C}^2$ is an \mathfrak{A}_5 -orbit Σ_{12} of length 12, because \mathcal{C}^2 is not contained in \mathcal{S}^1 . Similarly, we see that the intersection $\mathcal{S}^2 \cap \mathcal{C}^1$ is also an \mathfrak{A}_5 -orbit Σ'_{12} of length 12. These \mathfrak{A}_5 -orbits Σ_{12} and Σ'_{12} are different by Remark 3.10. Since B is the scheme theoretic intersection of the surfaces \mathcal{S}^1 and \mathcal{S}^2 , it must be singular at every point of $\Sigma_{12} \cup \Sigma'_{12}$. Denote by $\hat{\Sigma}_{12}$ and $\hat{\Sigma}'_{12}$ the preimages via ρ of the \mathfrak{A}_5 -orbits Σ_{12} and Σ'_{12} , respectively. Then \hat{B} is singular in every point of $\hat{\Sigma}'_{12}$.

The curve \hat{B} is smooth away of $\hat{\Sigma}'_{12}$, because its arithmetic genus is 21, and the surface $\hat{\mathcal{S}}^1$ does not contain \mathfrak{A}_5 -orbits of length less than 12. On the other hand, we have

$$\hat{B} \cap \hat{\mathcal{C}}^1 = \hat{\Sigma}_{12},$$

because $\hat{B} \cdot \hat{\mathcal{C}}^1 = 12$ and $\hat{\Sigma}_{12} \subset \hat{B}$. This shows that B is an irreducible curve whose only singularities are the points of $\Sigma_{12} \cup \Sigma'_{12}$, and each such point is an ordinary cusp of the curve B . In particular, the genus of the normalization of the curve B is 9. By [6, Lemma 5.1.5], this implies that B contains a unique \mathfrak{A}_5 -orbit of length 20. \square

The following classification of \mathfrak{A}_5 -invariant quartic surfaces in \mathbb{P}^3 was obtained in [4, Theorem 2.4].

Lemma 3.13. *The pencil \mathcal{P} contains two surfaces \mathcal{R}^1 and \mathcal{R}^2 with ordinary double singularities, such that the singular loci of \mathcal{R}^1 and \mathcal{R}^2 are \mathfrak{A}_5 -orbits of length 10. Every surface in \mathbb{P}^3 different from \mathcal{S}^1 , \mathcal{S}^2 , \mathcal{R}^1 and \mathcal{R}^2 is smooth.*

Proof. Let S be a surface in \mathcal{P} that is different from \mathcal{S}^1 and \mathcal{S}^2 . It follows from Lemma 3.3 that S is irreducible. Assume that S is singular.

We claim that S has isolated singularities. Indeed, suppose that S is singular along some \mathfrak{A}_5 -invariant curve Z . Taking a general plane section of S , we see that the degree of Z is at most three. Thus, one has either $Z = \mathcal{C}^1$ or $Z = \mathcal{C}^2$ by Lemma 3.8. Since neither of these curves is contained in the base locus of \mathcal{P} by Lemma 3.12, this would imply that either $S = \mathcal{S}^1$ or $S = \mathcal{S}^2$. The latter is not the case by assumption.

We see that the singularities of S are isolated. Hence, S contains at most two non-Du Val singular points by [23, Theorem 1] applied to the minimal resolution of singularities of the surface S . Since the set of all non-Du Val singular points of the surface S must be \mathfrak{A}_5 -invariant, we see that S has none of them by Lemma 3.2. Thus, all singularities of S are Du Val.

By [6, Lemma 6.7.3(iii)], the surface S has only ordinary double singularities, the set $\text{Sing}(S)$ consists of one \mathfrak{A}_5 -orbit, and

$$|\text{Sing}(S)| \in \{5, 6, 10, 12, 15\}.$$

Since \mathbb{P}^3 does not contain \mathfrak{A}_5 -orbits of lengths 5, 6, and 15 by Lemma 3.2, we see that $\text{Sing}(S)$ is either an \mathfrak{A}_5 -orbit of length 10 or an \mathfrak{A}_5 -orbit of length 12.

Suppose that the singular locus of S is an \mathfrak{A}_5 -orbit Σ_{12} of length 12. Then S does not contain other \mathfrak{A}_5 -orbits of length 12 by [6, Lemma 6.7.3(iv)]. Since \mathcal{C}^1 is not contained in the base locus of \mathcal{P} by Lemma 3.12, and \mathcal{C}^1 is contained in \mathcal{S}^1 , we see that $\mathcal{C}^1 \not\subset S$. Since

$$S \cdot \mathcal{C}^1 = 12$$

and Σ_{12} is the only \mathfrak{A}_5 -orbit of length at most 12 in $\mathcal{C}^1 \cong \mathbb{P}^1$, we have $S \cap \mathcal{C}^1 = \Sigma_{12}$. Thus,

$$12 = S \cdot \mathcal{C}^1 \geq \sum_{P \in \Sigma_{12}} \text{mult}_P(S) = 2|\Sigma_{12}| = 24,$$

which is absurd.

Therefore, we see that the singular locus of S is an \mathfrak{A}_5 -orbit Σ_{12} of length 10. Vice versa, if an \mathfrak{A}_5 -invariant quartic surface passes through an \mathfrak{A}_5 -orbit of length 10, then it is singular by [6, Lemma 6.7.1(ii)]. We know from Lemma 3.2 that there are exactly two \mathfrak{A}_5 -orbits of length 10 in \mathbb{P}^3 , and it follows from Lemma 3.12 that they are not contained in the base locus of \mathcal{P} . Thus there are two surfaces \mathcal{R}^1 and \mathcal{R}^2 that are singular exactly at the points of these two \mathfrak{A}_5 -orbits, respectively. The above argument shows that every surface in \mathcal{P} except \mathcal{S}^1 , \mathcal{S}^2 , \mathcal{R}^1 and \mathcal{R}^2 is smooth. \square

Keeping in mind Lemma 3.13, we will denote by \mathcal{R}^1 and \mathcal{R}^2 the two nodal surfaces contained in the pencil \mathcal{P} until the end of this section.

Lemma 3.14. *There is a unique \mathfrak{A}_5 -invariant quartic surface in \mathbb{P}^3 that contains the lines L_1, \dots, L_5 (respectively, the lines L'_1, \dots, L'_5). Moreover, this surface is smooth, and it does not contain the lines L'_1, \dots, L'_5 (respectively, L_1, \dots, L_5).*

Proof. Put $\mathcal{L} = \sum_{i=1}^5 L_i$ and $\mathcal{L}' = \sum_{i=1}^5 L'_i$. Corollary 2.1(ii) implies that the stabilizer in \mathfrak{A}_5 of a general point of L_1 is trivial. Therefore, there exists a surface $S \in \mathcal{P}$ that contains all lines L_1, \dots, L_5 . By Lemma 3.12 such surface S is unique.

We claim that $S \neq \mathcal{S}^1$. Indeed, all lines contained in \mathcal{S}^1 are tangent to the curve \mathcal{C}^1 , and there are no \mathfrak{A}_5 -orbits of length five in $\mathcal{C}^1 \cong \mathbb{P}^1$. Similarly, one has $S \neq \mathcal{S}^2$.

We claim that S is not one of the two nodal surfaces \mathcal{R}^1 and \mathcal{R}^2 contained in the pencil \mathcal{P} . Indeed, suppose that $S = \mathcal{R}^1$. Since the singular locus of \mathcal{R}^1 is an \mathfrak{A}_5 -orbit of length 10 by Lemma 3.13, we see that the lines L_1, \dots, L_5 are contained in the smooth locus of \mathcal{R}^1 by Corollary 3.5. On the other hand, one has $\mathcal{L}^2 = -10$ by Lemma 3.4. This means that $\text{rk Pic}(S)^{\mathfrak{A}_5} \geq 2$, which is impossible by [6, Lemma 6.7.3(i),(ii)].

We see that the surface S is different from \mathcal{R}^1 . A similar argument shows that S is different from \mathcal{R}^2 . Hence, S is smooth by Lemma 3.13.

Let us show that S does not contain the lines L'_1, \dots, L'_5 . Suppose that it does. By Lemma 3.4 one has

$$\mathcal{L} \cdot \mathcal{L} = \mathcal{L}' \cdot \mathcal{L}' = -10.$$

By [6, Lemma 6.7.1(i)], we have $\text{rk Pic}(S)^{\mathfrak{A}_5} = 2$. Let Π_S be the class of a plane section of S . Then the determinant of the matrix

$$\begin{pmatrix} \mathcal{L} \cdot \mathcal{L} & \mathcal{L} \cdot \mathcal{L}' & \Pi_S \cdot \mathcal{L} \\ \mathcal{L} \cdot \mathcal{L}' & \mathcal{L}' \cdot \mathcal{L}' & \Pi_S \cdot \mathcal{L}' \\ \Pi_S \cdot \mathcal{L} & \Pi_S \cdot \mathcal{L}' & \Pi_S \cdot \Pi_S \end{pmatrix} = \begin{pmatrix} -10 & 20 & 5 \\ 20 & -10 & 5 \\ 5 & 5 & 4 \end{pmatrix}$$

must vanish. This is a contradiction, because it equals 12.

Applying similar arguments, we see that the lines L'_1, \dots, L'_5 are contained in a unique \mathfrak{A}_5 -invariant quartic surface, this surface is smooth and does not contain the lines L_1, \dots, L_5 . \square

Remark 3.15. One can use the properties of the pencil \mathcal{P} to give an alternative proof of Lemma 3.7. Namely, we know from Lemma 3.14 that there are two (different) smooth \mathfrak{A}_5 -invariant quartic surfaces S and S' containing the lines L_1, \dots, L_5 and L'_1, \dots, L'_5 , respectively. By Lemma 3.12, the base locus of the pencil \mathcal{P} is an irreducible curve B that contains a unique \mathfrak{A}_5 -orbit Σ of length 20. By Corollary 3.5, this implies that Σ is contained in the union $L_1 \cup \dots \cup L_5$, because

$$B \cdot (L_1 + \dots + L_5) = 20$$

on the surface S . Similarly, we see that Σ is contained in $L'_1 \cup \dots \cup L'_5$. These facts together with Lemma 3.4 easily imply that the lines L_1, \dots, L_5 and L'_1, \dots, L'_5 form a double five configuration.

Now we will obtain some restrictions on low degree \mathfrak{A}_5 -invariant curves in \mathbb{P}^3 .

Lemma 3.16. *Let C be an irreducible \mathfrak{A}_5 -invariant curve in \mathbb{P}^3 of degree $d \leq 10$. Denote by g the genus of the normalization of the curve C . Then*

$$g \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)|.$$

Proof. Since \mathbb{U}_4 is an irreducible $2\mathfrak{A}_5$ -representation, the curve C is not contained in a plane in \mathbb{P}^3 . This implies that a stabilizer in \mathfrak{A}_5 of a general point of the curve C is trivial. In particular, the \mathfrak{A}_5 -orbit of a general point of C has length $|\mathfrak{A}_5| = 60$.

Let S be a surface in the pencil \mathcal{P} that passes through a general point of C . Then the curve C is contained in S , because otherwise one would have

$$60 \leq |S \cap C| \leq S \cdot C = 4d \leq 40,$$

which is absurd. Since the assertion of the lemma clearly holds for the twisted cubic curves \mathcal{C}^1 and \mathcal{C}^2 , we may assume that C is different from these two curves.

Suppose that $S = \mathcal{S}^1$. Let us use the notation of the proof of Lemma 3.12. Denote by \hat{C} the preimage of the curve C via ρ . Then \hat{C} is a divisor of bi-degree (a, b) for some non-negative integers a and b such that $d = 2a + b$. On the other hand, one has

$$|\hat{C} \cap \mathcal{C}^1| \leq \hat{C} \cdot \mathcal{C}^1 = a + b \leq 2a + b = d \leq 10,$$

which is impossible, since the curve $\mathcal{C}^1 \cong \mathcal{C}^1 \cong \mathbb{P}^1$ does not contain \mathfrak{A}_5 -orbits of length less than 12.

We see that $S \neq \mathcal{S}^1$. Similarly, we see that $S \neq \mathcal{S}^2$. By Lemma 3.13, either S is a smooth quartic $K3$ surface, or S is one of the surfaces \mathcal{R}^1 and \mathcal{R}^2 . Denote by Π_S a plane

section of S . Then

$$\det \begin{pmatrix} \Pi_S^2 & \Pi_S \cdot C \\ \Pi_S \cdot C & C^2 \end{pmatrix} = \det \begin{pmatrix} 4 & d \\ d & C^2 \end{pmatrix} = 4C^2 - d^2 \leq 0$$

by the Hodge index theorem.

Suppose that C is contained in the smooth locus of the surface S . Denote by $p_a(C)$ the arithmetic genus of the curve C . Then

$$C^2 = 2p_a(C) - 2.$$

by the adjunction formula. Thus, we get

$$p_a(C) \leq \frac{d^2}{8} + 1.$$

Since $g \leq p_a(C) - |\text{Sing}(C)|$, this implies the assertion of the lemma.

To complete the proof, we may assume that C contains a singular point of the surface S . By Lemma 3.13, this means that either $S = \mathcal{R}^1$ or $S = \mathcal{R}^2$. The singularities of the surface S are ordinary double points, and its singular locus is an \mathfrak{A}_5 -orbit of length 10. In particular, the curve C contains the whole singular locus of S . By [6, Theorem 6.7.1], one has $\text{Pic}(S)^{\mathfrak{A}_5} \cong \mathbb{Z}$. Since $\Pi_S^2 = 4$ and the self-intersection of any Cartier divisor on the surface S is even, we see that the group $\text{Pic}(S)^{\mathfrak{A}_5}$ is generated by Π_S .

Suppose that C is a Cartier divisor on S . Then either $C \sim \Pi_S$ or $C \sim 2\Pi_S$, because $d \leq 10$. Since the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(n)) \rightarrow H^0(\mathcal{O}_S(n\Pi_S))$$

is an isomorphism for $n \leq 3$, we conclude that there is an \mathfrak{A}_5 -invariant quadric in \mathbb{P}^3 . This is not the case by Lemma 3.3.

Therefore, we see that C is not a Cartier divisor on S . Since S has only ordinary double points, the divisor $2C$ is Cartier. Thus

$$2C \sim l\Pi_S,$$

for some odd positive integer l . Since

$$2d = 2C \cdot \Pi_S = l\Pi_S \cdot \Pi_S = 4l,$$

we see that $l = \frac{d}{2}$. In particular, d is even and $l \leq 5$.

Let $\theta: \tilde{S} \rightarrow S$ be the minimal resolution of singularities of the surface S . Denote by \tilde{C} the proper transform of the curve C on the surface \tilde{S} , and denote by $\Theta_1, \dots, \Theta_{10}$ the exceptional curves of θ . Then

$$2\tilde{C} \sim \theta^*(l\Pi_S) - m \sum_{i=1}^{10} \Theta_i,$$

for some positive integer m . Moreover, m is odd, because C is not a Cartier divisor. We have

$$4\tilde{C}^2 = \Pi_S^2 l^2 - 20m^2 = 4l^2 - 20m^2,$$

which implies that $\tilde{C}^2 = l^2 - 5m^2$. Since \tilde{C}^2 is even, m is odd and $l \leq 5$, we see that either $l = 3$ or $l = 5$.

Denote by $p_a(\tilde{C})$ the arithmetic genus of the curve \tilde{C} . Then

$$l^2 - 5m^2 = \tilde{C}^2 = 2p_a(\tilde{C}) - 2.$$

by the adjunction formula. In particular, we have

$$25 - 5m^2 \geq l^2 - 5m^2 \geq -2,$$

so that $l \in \{3, 5\}$ and $m = 1$. The latter means that C is smooth at every point of $\text{Sing}(S)$, so that

$$|\text{Sing}(\tilde{C})| = |\text{Sing}(C)|.$$

If $l = \frac{d}{2} = 3$, then $p_a(\tilde{C}) = 3$. This gives

$$g \leq p_a(\tilde{C}) - |\text{Sing}(\tilde{C})| = 3 - |\text{Sing}(C)| \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)|.$$

Similarly, if $l = \frac{d}{2} = 5$, then $p_a(\tilde{C}) = 11$. This gives

$$g \leq p_a(\tilde{C}) - |\text{Sing}(\tilde{C})| = 11 - |\text{Sing}(C)| \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)|.$$

□

Recall from [6, Lemma 5.4.1] that there exists a unique smooth irreducible curve of genus 4 with a faithful action of the group \mathfrak{A}_5 . This curve is known as the Bring's curve. Its canonical model is a complete intersection of a quadric and a cubic in a three-dimensional projective space. However, this sextic curve does not appear in our $\mathbb{P}^3 = \mathbb{P}(\mathbb{U}_4)$ by

Lemma 3.17. *Let C be a smooth irreducible \mathfrak{A}_5 -invariant curve in \mathbb{P}^3 of degree $d \leq 6$ and genus g . Then $g \neq 4$.*

Proof. Suppose that $g = 4$. Denote by Π_C the plane section of the curve C . Then

$$h^0(\mathcal{O}_C(\Pi_C)) = d - 3 + h^0(\mathcal{O}_C(K_C - \Pi_C))$$

by the Riemann–Roch theorem. Since C is not contained in a plane, this implies that $\Pi_C \sim K_C$. Therefore, the projective space \mathbb{P}^3 is identified with a projectivization of an \mathfrak{A}_5 -representation $H^0(\mathcal{O}_C(K_C))^\vee$, i.e. of a representation of the group $2.\mathfrak{A}_5$ where the center of $2.\mathfrak{A}_5$ acts trivially. The latter is not the case by construction of \mathbb{U}_4 . □

Lemma 3.18. *Let C be an irreducible smooth \mathfrak{A}_5 -invariant curve in \mathbb{P}^3 of degree $d = 10$ and genus g . Then $g \neq 10$.*

Proof. Suppose that $g = 10$. By Lemma 3.12, the base locus of the pencil \mathcal{P} is an irreducible curve B of degree 16. In particular, there exists a surface $S \in \mathcal{P}$ that does not contain C . Thus, the intersection $S \cap C$ is an \mathfrak{A}_5 -invariant set that consists of

$$C \cdot S = 4d = 40$$

points (counted with multiplicities). On the other hand, by [6, Lemma 5.1.5], any \mathfrak{A}_5 -orbit in C has length 12, 30, or 60. □

4. LARGE SUBGROUPS OF \mathfrak{S}_6

In this section we collect some auxiliary results about the groups \mathfrak{S}_6 , \mathfrak{A}_6 and \mathfrak{S}_5 . We start with recalling some general properties of the group \mathfrak{A}_6 .

Remark 4.1 (see e.g. [8, p. 4]). Let Γ be a proper subgroup of \mathfrak{A}_6 such that the index of Γ is at most 15. Then Γ is isomorphic either to \mathfrak{A}_5 , or to F_{36} , or to \mathfrak{S}_4 . In particular, if \mathfrak{A}_6 acts transitively on the set of $r < 15$ elements, then either $r = 6$ or $r = 10$.

We will need the following result about possible actions of the group \mathfrak{A}_6 on curves of small genera (cf. [5, Theorem 2.18] and [6, Lemma 5.1.5]).

Lemma 4.2. *Suppose that C is a smooth irreducible curve of genus $g \leq 15$ with a non-trivial action of the group \mathfrak{A}_6 . Then $g = 10$.*

Proof. Let $\Omega \subset C$ be an \mathfrak{A}_6 -orbit. Then a stabilizer of a point in Ω is a cyclic subgroup of \mathfrak{A}_6 , which implies that

$$|\Omega| \in \{72, 90, 120, 180, 360\}.$$

From the classification of finite subgroups of $\text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2(\mathbb{C})$ we know that $g \neq 0$. Also, it follows from the non-solvability of the group \mathfrak{A}_6 that $g \neq 1$.

Put $\bar{C} = C/\mathfrak{A}_6$. Then \bar{C} is a smooth curve. Let \bar{g} be the genus of the curve \bar{C} . The Riemann–Hurwitz formula gives

$$2g - 2 = 360(2\bar{g} - 2) + 180a_{180} + 240a_{120} + 270a_{90} + 288a_{72},$$

where a_k is the number of \mathfrak{A}_6 -orbits in C of length k .

Since $a_k \geq 0$ and $2 \leq g \leq 15$, one has $\bar{g} = 0$. Thus, we obtain

$$2g - 2 = -720 + 180a_{180} + 240a_{120} + 270a_{90} + 288a_{72}.$$

Going through the values $2 \leq g \leq 15$, and solving this equation case by case we see that the only possibility is $g = 10$. \square

We proceed by recalling some general properties of the group \mathfrak{S}_5 .

Remark 4.3 (see e.g. [8, p. 2]). Let Γ be a proper subgroup of \mathfrak{S}_5 such that the index of Γ is less than 12. Then Γ is isomorphic either to \mathfrak{A}_5 , or to \mathfrak{S}_4 , or to F_{20} , or to \mathfrak{A}_4 , or to D_{12} . In particular, if \mathfrak{S}_5 acts transitively on the set of $r < 12$ elements, then $r \in \{2, 5, 6, 10\}$.

Lemma 4.4. *The group \mathfrak{S}_5 cannot act faithfully on a smooth irreducible curve of genus 5.*

Proof. Suppose that C is a curve of genus 5 with a faithful action of \mathfrak{S}_5 . Considering the action of the subgroup $\mathfrak{A}_5 \subset \mathfrak{S}_5$ on C and applying [6, Lemma 5.4.3], we see that C is hyperelliptic. This gives a natural homomorphism

$$\theta: \mathfrak{S}_5 \rightarrow \text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2(\mathbb{C})$$

whose kernel is either trivial or isomorphic to μ_2 . Thus θ is injective, which gives a contradiction. \square

Now we will prove some auxiliary facts about actions of the groups \mathfrak{S}_6 , \mathfrak{A}_6 and \mathfrak{S}_5 on the four-dimensional projective space.

Remark 4.5. The group \mathfrak{S}_6 has exactly four irreducible five-dimensional representations (see e.g. [8, p. 5]). Starting from one of them, one more can be obtained by a twist by an outer automorphism of \mathfrak{S}_6 , and two remaining ones are obtained from these two by a tensor product with the sign representation. Although these four representations are not isomorphic, the images of \mathfrak{S}_6 in $\text{PGL}_5(\mathbb{C})$ under them are the same. Every irreducible five-dimensional representation of \mathfrak{S}_6 restricts to an irreducible representation of the subgroup $\mathfrak{A}_6 \subset \mathfrak{S}_6$, and restricts to an irreducible representation of the *some* of the subgroups $\mathfrak{S}_5 \subset \mathfrak{S}_6$. The group \mathfrak{A}_6 has exactly two irreducible five-dimensional representations, each of them arising this way (see e.g. [8, p. 5]). Similarly, the group \mathfrak{S}_5 has exactly two irreducible five-dimensional representations, each of them arising this

way (see e.g. [8, p. 2]). Note also that every five-dimensional representation of a group \mathfrak{A}_6 or \mathfrak{S}_5 that does not contain one-dimensional subrepresentations is irreducible.

Let \mathbb{V}_5 be an irreducible five-dimensional representation of the group \mathfrak{S}_6 . Put $\mathbb{P}^4 = \mathbb{P}(\mathbb{V}_5)$. Keeping in mind Remark 4.5, we see that the image of the corresponding homomorphism \mathfrak{S}_6 to $\mathrm{PGL}_5(\mathbb{C})$ is the same for any choice of \mathbb{V}_5 , and thus the \mathfrak{S}_6 -orbits and \mathfrak{S}_6 -invariant hypersurfaces in \mathbb{P}^4 do not depend on \mathbb{V}_5 either.

Remark 4.5 implies that there are six linear forms x_0, \dots, x_5 on \mathbb{P}^4 that are permuted by the group \mathfrak{S}_6 (cf. Sections 1 and 2). Indeed, up to a twist by an outer automorphism of \mathfrak{S}_6 and a tensor product with the sign representation, \mathbb{V}_5 is a subrepresentation of the six-dimensional representation \mathbb{W} of \mathfrak{S}_6 , so that one can take restrictions of the natural coordinates in \mathbb{W} to be these linear forms. Let Q be the three-dimensional quadric in \mathbb{P}^4 given by equation

$$(4.6) \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0.$$

The quadric Q is smooth and \mathfrak{S}_6 -invariant. Note also that equation (1.1) makes sense in our \mathbb{P}^4 .

We will use the notation introduced above until the end of the paper.

Lemma 4.7. *Let Γ be either the group \mathfrak{S}_6 , or its subgroup \mathfrak{A}_6 , or a subgroup \mathfrak{S}_5 of \mathfrak{S}_6 such that \mathbb{V}_5 is an irreducible representation of Γ . Then the only Γ -invariant quadric threefold in \mathbb{P}^4 is the quadric Q . Similarly, every (reduced) Γ -invariant quartic threefold in \mathbb{P}^4 is given by equation (1.1) for some $t \in \mathbb{C}$.*

Proof. Apply Corollary 2.5. □

By a small abuse of notation we will refer to the points in \mathbb{P}^4 using x_i as if they were homogeneous coordinates, i.e. a point in \mathbb{P}^4 will be encoded by a ratio of six linear forms x_i . As in Section 1, let Σ_6 and Σ_{10} are the \mathfrak{S}_6 -orbits of the points $[-5 : 1 : 1 : 1 : 1 : 1]$ and $[-1 : -1 : -1 : 1 : 1 : 1]$, respectively. Looking at equation (4.6), we obtain

Corollary 4.8. *The quadric Q does not contain the \mathfrak{S}_6 -orbits Σ_6 and Σ_{10} .*

Now we will have a look at the action of the group \mathfrak{A}_6 on \mathbb{P}^4 . Note that \mathbb{V}_5 is an irreducible \mathfrak{A}_6 -representation by Remark 4.5.

Lemma 4.9. *There are no \mathfrak{A}_6 -orbits of length less than six in \mathbb{P}^4 . Moreover, the only \mathfrak{A}_6 -orbit of length six in \mathbb{P}^4 is Σ_6 .*

Proof. The only subgroup of \mathfrak{A}_6 of index less than six is \mathfrak{A}_6 itself (cf. Remark 4.1), so that the first assertion of the lemma follows from irreducibility of the \mathfrak{A}_6 -representation \mathbb{V}_5 . Also, the only subgroups of \mathfrak{A}_6 of index six are \mathfrak{A}_5^{st} and \mathfrak{A}_5^{nst} , so that the second assertion of the lemma also follows from Corollary 2.3. □

Lemma 4.10. *Let X be an \mathfrak{A}_6 -invariant quartic threefold in \mathbb{P}^4 that contains an \mathfrak{A}_6 -orbit of length at most six. Then $X = X_{\frac{7}{10}}$.*

Proof. By Lemma 4.7, one has $X = X_t$ for some $t \in \mathbb{C}$, and by Lemma 4.9 the \mathfrak{A}_6 -orbit Σ_6 is contained in X_t . Since Σ_6 is not contained in the quadric Q by Corollary 4.8, we see that there is a unique $t \in \mathbb{C}$ such that Σ_6 is contained in a quartic given by equation (1.1). Therefore, we conclude that $t = \frac{7}{10}$. □

Now we will make a couple of observations about the action of the group \mathfrak{S}_5 on \mathbb{P}^4 . We choose \mathfrak{S}_5 to be a subgroup of \mathfrak{S}_6 such that \mathbb{V}_5 is an irreducible \mathfrak{S}_5 -representation (cf. Remark 4.5 and Corollary 2.3).

Lemma 4.11. *Let $P \in \mathbb{P}^4$ be a point such that its stabilizer in \mathfrak{S}_5 contains a subgroup isomorphic to D_{12} . Then the \mathfrak{S}_5 -orbit of P is Σ_{10} .*

Proof. By Corollary 2.3(iii), the point in \mathbb{P}^4 fixed by a subgroup $D_{12} \subset \mathfrak{S}_5$ is unique. On the other hand, it is straightforward to check that a stabilizer in \mathfrak{S}_5 of a point of Σ_{10} contains a subgroup isomorphic to D_{12} . It remains to notice that the latter stabilizer is actually isomorphic to D_{12} , since the only subgroups of \mathfrak{S}_5 that contain D_{12} are D_{12} and \mathfrak{S}_5 itself, while \mathfrak{S}_5 has no fixed points on \mathbb{P}^4 . \square

Lemma 4.12. *Let X be an \mathfrak{S}_5 -invariant quartic threefold in \mathbb{P}^4 that contains Σ_{10} . Then $X = X_{\frac{1}{6}}$.*

Proof. By Lemma 4.7, one has $X = X_t$ for some $t \in \mathbb{C}$. Since Σ_{10} is not contained in the quadric Q by Corollary 4.8, we see that there is a unique $t \in \mathbb{C}$ such that Σ_{10} is contained in a quartic given by equation (1.1). Therefore, we conclude that $t = \frac{1}{6}$. \square

5. RATIONALITY OF THE QUARTIC THREEFOLD $X_{\frac{7}{10}}$

In this section we will construct an explicit \mathfrak{A}_6 -equivariant birational map $\mathbb{P}^3 \dashrightarrow X_{\frac{7}{10}}$. Implicitly, the construction of this map first appeared in the proof of [5, Theorem 1.20]. Here we will present a much simplified proof of its existence.

We identify \mathbb{P}^3 with the projectivization $\mathbb{P}(\mathbb{U}_4)$, where \mathbb{U}_4 is the restriction of the four-dimensional irreducible representation of the group $2.\mathfrak{S}_6$ introduced in Section 2 to the subgroup $2.\mathfrak{A}_6$. By Corollary 2.1(i), the $2.\mathfrak{A}_6$ -representation \mathbb{U}_4 is irreducible.

Lemma 5.1. *There are no \mathfrak{A}_6 -invariant surfaces of odd degree in \mathbb{P}^3 , and no \mathfrak{A}_6 -invariant pencils of surfaces of odd degree in \mathbb{P}^3 . Moreover, there are no \mathfrak{A}_6 -invariant quadric and quartic surfaces in \mathbb{P}^3 .*

Proof. Recall that the only one-dimensional representation of the group $2.\mathfrak{A}_6$ is the trivial representation. Therefore, any \mathfrak{A}_6 -invariant surface of odd degree d in \mathbb{P}^3 gives rise to a trivial $2.\mathfrak{A}_6$ -subrepresentation in $R_d = \text{Sym}^d(\mathbb{U}_4)$. On the other hand, the non-trivial central element z of $2.\mathfrak{A}_6$ acts on R_d by a scalar matrix with diagonal entries equal to -1 , which shows that R_d does not contain trivial $2.\mathfrak{A}_6$ -representations. Also, since the only two-dimensional representation of $2.\mathfrak{A}_6$ is the sum of two trivial representations, this implies that there are no \mathfrak{A}_6 -invariant pencils of surfaces of odd degree in \mathbb{P}^3 .

The last assertion of the lemma follows from Corollary 2.4(i),(ii). \square

Lemma 5.2. *Let Ω be an \mathfrak{A}_6 -orbit in \mathbb{P}^3 . Then $|\Omega| \geq 16$.*

Proof. Lemma 5.1 implies that there are no \mathfrak{A}_6 -orbits of odd length in \mathbb{P}^3 . Thus, if Ω is an \mathfrak{A}_6 -orbit in \mathbb{P}^3 of length at most 15, then by Remark 4.1 a stabilizer of its general point is isomorphic either to \mathfrak{A}_5 or to F_{36} . Both of these cases are impossible by Corollary 2.1. \square

Actually, the minimal degree of an \mathfrak{A}_6 -invariant surface in \mathbb{P}^3 equals 8 (see [5, Lemma 3.7]), and the minimal length of an \mathfrak{A}_6 -orbit in \mathbb{P}^3 equals 36 (see [5, Lemma 3.8]), but we will not need this here.

Lemma 5.3 (cf. [5, Lemma 4.26]). *Let C be a smooth irreducible \mathfrak{A}_6 -invariant curve of degree 9 and genus g in \mathbb{P}^3 . Then $g \neq 10$.*

Proof. Suppose that $g = 10$. Then it follows from [13, Example 6.4.3] that either C is contained in a unique quadric surface in \mathbb{P}^3 , or C is a complete intersection of two cubic surfaces in \mathbb{P}^3 . The former case is impossible, since there are no \mathfrak{A}_6 -invariant quadrics in \mathbb{P}^3 by Lemma 5.1. The latter case is impossible, because there are no \mathfrak{A}_6 -invariant pencils of cubic surfaces in \mathbb{P}^3 by Lemma 5.1. \square

Recall that the group \mathfrak{A}_6 contains six standard subgroups isomorphic to \mathfrak{A}_5 and six non-standard subgroups isomorphic to \mathfrak{A}_5 (see the conventions made in Section 2). Denote the former ones by H'_1, \dots, H'_6 , and denote the latter ones by H_1, \dots, H_6 . By Corollary 2.1(ii), each group H'_i leaves invariant two lines L_i^1 and L_i^2 in \mathbb{P}^3 . Note that each group H_i permutes transitively the lines L_1^1, \dots, L_6^1 (respectively, L_1^2, \dots, L_6^2).

Put $\mathcal{L}^1 = L_1^1 + \dots + L_6^1$ and $\mathcal{L}^2 = L_1^2 + \dots + L_6^2$. Then the curves \mathcal{L}^1 and \mathcal{L}^2 are \mathfrak{A}_6 -invariant, and the curve $\mathcal{L}^1 + \mathcal{L}^2$ is \mathfrak{S}_6 -invariant.

Lemma 5.4. *The lines L_1^1, \dots, L_6^1 (respectively, the lines L_1^2, \dots, L_6^2) are pairwise disjoint. Moreover, the curves \mathcal{L}^1 and \mathcal{L}^2 are disjoint.*

Proof. We use an argument similar to one in the proof of Lemma 3.4. Suppose that some of the lines L_1^1, \dots, L_6^1 have a common point. Since the action of \mathfrak{A}_6 on the set $\{L_1^1, \dots, L_6^1\}$ is doubly transitive, this implies that any two of the lines L_1^1, \dots, L_6^1 have a common point. Therefore, either all lines L_1^1, \dots, L_6^1 are coplanar, or all of them pass through one point. Both of these cases are impossible since \mathbb{U}_4 is an irreducible $2\mathfrak{A}_6$ -representation (see Corollary 2.1(i)). Therefore, the lines L_1^1, \dots, L_6^1 are pairwise disjoint. A similar argument applies to the lines L_1^2, \dots, L_6^2 .

Suppose that some of the lines L_1^1, \dots, L_6^1 , say, L_1^1 , intersects some of the lines L_1^2, \dots, L_6^2 . Since the lines L_1^1 and L_2^1 are disjoint by construction, we may assume that L_1^1 intersects L_2^2 . Since the stabilizer $H'_1 \subset \mathfrak{A}_6$ of L_1^1 acts transitively on the lines L_2^2, \dots, L_6^2 , we conclude that all five lines L_2^2, \dots, L_6^2 intersect L_1^1 . Therefore, the line L_1^1 contains a subset of at most five points that is invariant with respect to the group $H'_1 \cong \mathfrak{A}_5$, which is a contradiction. Thus, \mathcal{L}^1 and \mathcal{L}^2 are disjoint. \square

Lemma 5.5. *Let C be an \mathfrak{A}_6 -invariant curve in \mathbb{P}^3 of degree $d \leq 10$. Then either $C = \mathcal{L}^1$ or $C = \mathcal{L}^2$.*

Proof. Suppose first that C is reducible. We may assume that \mathfrak{A}_6 permutes the irreducible components of C transitively. Thus, C has either 6 or 10 irreducible components by Remark 4.1, and these irreducible components are lines. By Remark 4.1 and Corollary 2.1 the latter case is impossible, and in the former case one has either $C = \mathcal{L}^1$ or $C = \mathcal{L}^2$.

Therefore, we assume that the curve C is irreducible. Let g be the genus of the normalization of the curve C . We have

$$(5.6) \quad g \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)| \leq 13 - |\text{Sing}(C)|$$

by Lemma 3.16. This implies that the curve C is smooth, because \mathbb{P}^3 does not contain \mathfrak{A}_6 -orbits of length less than 16 by Lemma 5.2.

If $d \leq 8$, then (5.6) gives $g \leq 9$. This is impossible by Lemma 4.2

If $d = 9$, then (5.6) gives $g \leq 11$, so that $g = 10$ by Lemma 4.2. This is impossible by Lemma 5.3.

Therefore, we see that $d = 10$. Thus, (5.6) gives $g \leq 13$, so that $g = 10$ by Lemma 4.2. The latter is impossible by Lemma 3.18. \square

Denote by \mathcal{M} the linear system on \mathbb{P}^3 consisting of all quartic surfaces passing through the lines L_1^1, \dots, L_6^1 . Then \mathcal{M} is not empty. In fact, its dimension is at least four by parameter count. Moreover, the linear system \mathcal{M} does not have base components by Lemma 5.1.

Lemma 5.7. *The base locus of \mathcal{M} does not contain curves except the lines L_1^1, \dots, L_6^1 . Moreover, a general surface in \mathcal{M} is smooth at a general point of each of the lines L_1^1, \dots, L_6^1 .*

Proof. Denote by Z the union of the curves that are contained in the base locus of \mathcal{M} and are different from the lines L_1^1, \dots, L_6^1 . Then Z is a (possibly empty) \mathfrak{A}_6 -invariant curve. Denote its degree by d . Pick two general surfaces M_1 and M_2 in \mathcal{M} . Then

$$M_1 \cdot M_2 = Z + m\mathcal{L}^1 + \Delta,$$

where m is a positive integer, and Δ is an effective one-cycle on \mathbb{P}^3 that contains none of the lines L_1^1, \dots, L_6^1 . Note that Δ may contain irreducible components of the curve Z . Let Π be a plane in \mathbb{P}^3 . Then

$$16 = \Pi \cdot M_1 \cdot M_2 = \Pi \cdot Z + m\Pi \cdot \mathcal{L}^1 + \Pi \cdot \Delta = d + 6m + \Pi \cdot \Delta \leq d + 6m,$$

which implies that $m \leq 2$ and $d \leq 10$. By Lemma 5.5, we have $d = 0$, so that Z is empty. Since

$$2 \geq m \geq \text{mult}_{L_i^1}(M_1)\text{mult}_{L_i^1}(M_2),$$

we see that a general surface in \mathcal{M} is smooth at a general point of L_i^1 . \square

Let $\alpha: U \rightarrow \mathbb{P}^3$ be a blow up along the lines L_1^1, \dots, L_6^1 . Then $-K_U^3 = 4$, and the action of \mathfrak{A}_6 lifts to U . Denote by E_1, \dots, E_6 the α -exceptional surfaces that are mapped to L_1^1, \dots, L_6^1 , respectively.

Lemma 5.8. *The action of the stabilizer $H'_i \cong \mathfrak{A}_5$ in \mathfrak{A}_6 of the line L_i^1 on the surface $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ is twisted diagonal, i.e., E_i is identified with $\mathbb{P}(\mathbb{U}_2) \times \mathbb{P}(\mathbb{U}'_2)$, where \mathbb{U}_2 and \mathbb{U}'_2 are different two-dimensional irreducible representations of the group $2.\mathfrak{A}_5$.*

Proof. This follows from Corollary 2.1(ii). \square

Let us denote by \mathcal{M}_U the proper transform of the linear system \mathcal{M} on the threefold U . Then $\mathcal{M}_U \sim -K_U$ by Lemma 5.7.

Lemma 5.9. *The linear system \mathcal{M}_U is base point free.*

Proof. Let us first show that \mathcal{M}_U is free from base curves. Suppose that the base locus of the linear system \mathcal{M}_U contains some curves. Then each of these curves is contained in some of the α -exceptional surfaces by Lemma 5.7. Denote by Z the union of all such curves that are contained in E_1 . Then Z is an H'_1 -invariant curve. For some non-negative integers a and b , one has

$$Z \sim as + bl,$$

where s is a section of the natural projection $E_1 \rightarrow L_1^1$ such that $s^2 = 0$ on E_1 , and l is a fiber of this projection. On the other hand, we have

$$\mathcal{M}_U|_{E_1} \sim -K_U|_{E_1} \sim s + 3l.$$

This gives $a \leq 1$ and $b \leq 3$. Since the action of H'_1 on the surface E_1 is twisted diagonal by Lemma 5.8, the latter is impossible by [6, Lemma 6.4.2(i)] and [6, Lemma 6.4.11(o)].

We see that \mathcal{M}_U is free from base curves. Since $\mathcal{M}_U \sim -K_U$, the linear system \mathcal{M}_U cannot have more than $-K_U^3 = 4$ base points. By Lemma 5.2, this implies that \mathcal{M}_U is base point free. \square

Corollary 5.10. *The base locus of the linear system \mathcal{M} consists of the lines L_1^1, \dots, L_6^1 .*

By Lemma 5.9, the divisor $-K_U$ is nef. Since $-K_U^3 = 4$, it is also big. Thus, we have

$$h^1(\mathcal{O}_U(-K_U)) = h^2(\mathcal{O}_U(-K_U)) = 0$$

by the Kawamata–Viehweg vanishing theorem (see [17]). Hence, the Riemann–Roch formula gives

$$(5.11) \quad h^0(\mathcal{O}_U(-K_U)) = 5.$$

In particular, we see that $|-K_U| = \mathcal{M}_U$.

Lemma 5.12. *The \mathfrak{A}_6 -representation $H^0(\mathcal{O}_U(-K_U))$ is irreducible.*

Proof. By Lemma 5.1, there are no \mathfrak{A}_6 -invariant quartic surfaces in \mathbb{P}^3 . This implies that $H^0(\mathcal{O}_U(-K_U))$ does not contain one-dimensional subrepresentations. Hence it is irreducible by Remark 4.5. \square

Lemma 5.9 together with (5.11) implies that there is an \mathfrak{A}_6 -equivariant commutative diagram

$$(5.13) \quad \begin{array}{ccc} & U & \\ \alpha \swarrow & & \searrow \beta \\ \mathbb{P}^3 & \dashrightarrow & \mathbb{P}^4, \\ & \phi & \end{array}$$

where ϕ is a rational map given by \mathcal{M} , and β is a morphism given by the anticanonical linear system $|-K_U|$. By Lemma 5.12 the projective space \mathbb{P}^4 in (5.13) is a projectivization of an irreducible \mathfrak{A}_6 -representation.

Recall from Lemma 3.8 that \mathbb{P}^3 contains exactly two H_1 -invariant twisted cubic curves \mathcal{C}_1^1 and \mathcal{C}_1^2 .

Lemma 5.14. *The curve \mathcal{L}^1 intersects exactly one curve among \mathcal{C}_1^1 and \mathcal{C}_1^2 . Moreover, each line among L_1^1, \dots, L_6^1 contains two points of this intersection. Similarly, the curve \mathcal{L}^2 intersects exactly one curve among \mathcal{C}_1^1 and \mathcal{C}_1^2 , and this curve is different from the one that intersects \mathcal{L}^1 .*

Proof. By Remark 3.1, the stabilizer in H_1 of the curve L_1^1 is isomorphic to D_{10} , and thus it has an orbit of length 2 on L_1^1 . Thus, the curve \mathcal{L}^1 contains an H_1 -orbit Σ_{12}^1 of length 12 by Lemma 3.2. Similarly, the curve \mathcal{L}^2 contains an H_1 -orbit Σ_{12}^2 of length 12. By Lemma 5.4, one has $\Sigma_{12}^1 \neq \Sigma_{12}^2$. Moreover, Σ_{12}^1 and Σ_{12}^2 are the only H_1 -orbits in \mathbb{P}^3 of length 12 by Lemma 3.2. Since \mathcal{C}_1^1 and \mathcal{C}_1^2 are disjoint by Remark 3.10, and each of them contains an H_1 -orbit of length 12, we see that either $\Sigma_{12}^1 \subset \mathcal{C}_1^1$ and $\Sigma_{12}^2 \subset \mathcal{C}_1^2$, or $\Sigma_{12}^2 \subset \mathcal{C}_1^1$ and $\Sigma_{12}^1 \subset \mathcal{C}_1^2$. Since a line cannot have more than two common points with a twisted cubic, this also implies the last assertion of the lemma. \square

Without loss of generality, we may assume that the curve \mathcal{L}^1 intersects \mathcal{C}_1^1 , and the curve \mathcal{L}^2 intersects \mathcal{C}_1^2 . Let $\mathcal{C}_1^1, \dots, \mathcal{C}_6^1$ be the \mathfrak{A}_6 -orbit of the curve \mathcal{C}_1^1 , and let $\mathcal{C}_1^2, \dots, \mathcal{C}_6^2$ be the \mathfrak{A}_6 -orbit of the curve \mathcal{C}_1^2 . By Lemma 3.8, the curves \mathcal{C}_i^1 and \mathcal{C}_i^2 are the only twisted cubic curves in \mathbb{P}^3 that are H_i -invariant. By Lemma 5.14, we have

Corollary 5.15. *Every twisted cubic curve \mathcal{C}_i^1 intersects each line among L_1^1, \dots, L_6^1 by two points. Similarly, every twisted cubic curve \mathcal{C}_i^2 intersects each line among L_1^2, \dots, L_6^2 by two points.*

Denote by $\tilde{\mathcal{C}}_1^1, \dots, \tilde{\mathcal{C}}_6^1$ the proper transforms on U of the curves $\mathcal{C}_1^1, \dots, \mathcal{C}_6^1$, respectively.

Lemma 5.16. *One has $-K_U \cdot \tilde{\mathcal{C}}_1^1 = \dots = -K_U \cdot \tilde{\mathcal{C}}_6^1 = 0$.*

Proof. This follows from Corollary 5.15. \square

We see that each curve $\tilde{\mathcal{C}}_i^1$ is contracted by β to a point. Since the \mathfrak{A}_6 -orbit of $\tilde{\mathcal{C}}_1^1$ consists of six curves, we also obtain the following.

Corollary 5.17. *The image of the morphism β contains an \mathfrak{A}_6 -orbit of length at most six.*

Since $-K_U^3 = 4$, the image of β is either an \mathfrak{A}_6 -invariant quartic threefold or an \mathfrak{A}_6 -invariant quadric threefold. Using results of [24], one can show that the latter case is impossible. However, this immediately follows from Corollary 5.17. Indeed, an \mathfrak{A}_6 -orbit of length at most six cannot be contained in the \mathfrak{A}_6 -invariant quadric by Corollary 4.8 and Lemma 4.9.

Corollary 5.18. *The morphism β is birational onto its image, and its image is a quartic threefold.*

Now Lemma 4.10 implies that the image of β is the quartic $X_{\frac{7}{10}}$. This proves

Corollary 5.19. *The threefold $X_{\frac{7}{10}}$ is rational.*

Let us conclude this section by recalling two related results proved in [5, §4]. The commutative diagram (5.13) can be extended to an \mathfrak{A}_6 -equivariant commutative diagram

$$(5.20) \quad \begin{array}{ccccc} U & \xrightarrow{\quad \rho \quad} & U & & \\ \alpha \searrow & \gamma \searrow & & \gamma \swarrow & \alpha \swarrow \\ & X_{\frac{7}{10}} & \xrightarrow{\quad \sigma \quad} & X_{\frac{7}{10}} & \\ \phi \nearrow & & & & \phi \nearrow \\ \mathbb{P}^3 & \xrightarrow{\quad \psi \quad} & \mathbb{P}^3 & & \end{array}$$

Here σ is an automorphism of the quartic threefold $X_{\frac{7}{10}}$ given by an odd permutation in \mathfrak{S}_6 acting on \mathbb{P}^4 , cf. Remark 4.5. The birational map ρ is a composition of Atiyah flops in 36 curves contracted by γ , and the birational map ψ is not regular.

The diagram (5.20) is a so-called \mathfrak{A}_6 -Sarkisov link. The subgroup $\mathfrak{A}_6 \subset \text{Aut}(\mathbb{P}^3)$ together with $\psi \in \text{Bir}^{\mathfrak{A}_6}(\mathbb{P}^3)$ generates a subgroup isomorphic to \mathfrak{S}_6 . Moreover, the subgroup

$$\text{Aut}^{\mathfrak{A}_6}(\mathbb{P}^3) \subset \text{Bir}^{\mathfrak{A}_6}(\mathbb{P}^3)$$

is also isomorphic to \mathfrak{S}_6 . By [5, Theorem 1.24], the whole group $\text{Bir}^{\mathfrak{A}_6}(\mathbb{P}^3)$ is a free product of these two copies of \mathfrak{S}_6 amalgamated over the original \mathfrak{A}_6 .

6. RATIONALITY OF THE QUARTIC THREEFOLD $X_{\frac{1}{6}}$

In this section we will construct an explicit \mathfrak{S}_5 -equivariant birational map $\mathbb{P}^3 \dashrightarrow X_{\frac{1}{6}}$. We identify \mathbb{P}^3 with the projectivization $\mathbb{P}(\mathbb{U}_4)$, where \mathbb{U}_4 is the restriction of the four-dimensional irreducible representation of the group $2.\mathfrak{S}_6$ introduced in Section 2 to a subgroup $2.\mathfrak{S}_5^{nst}$, and denote the latter subgroup simply by $2.\mathfrak{S}_5$. By Corollary 2.1(i), the $2.\mathfrak{S}_5$ -representation \mathbb{U}_4 is irreducible.

Lemma 6.1. *Let Ω be an \mathfrak{S}_5 -orbit in \mathbb{P}^3 . Then $|\Omega| \geq 12$.*

Proof. Apply Remark 4.3 together with Corollary 2.1. \square

Lemma 6.2. *Let C be an \mathfrak{S}_5 -invariant curve in \mathbb{P}^3 of degree d . Then $d \geq 6$.*

Proof. Suppose that $d \leq 5$. To start with, assume that C is reducible and denote by r the number of its irreducible components. We may assume that \mathfrak{S}_5 permutes the irreducible components of C transitively. Thus, either $r = 2$ or $r = 5$ by Remark 4.3. If $r = 5$, the irreducible components of C are lines, so that this case is impossible by Remark 4.3 and Corollary 2.1(i). Hence, we have $r = 2$, and the stabilizer of each of the two irreducible components C_1 and C_2 of C is the subgroup $\mathfrak{A}_5 \subset \mathfrak{S}_5$. Moreover, in this case one has

$$\deg(C_1) = \deg(C_2) \leq 2,$$

which is impossible by Lemma 3.8.

Therefore, we assume that the curve C is irreducible. Let g be the genus of the normalization of the curve C . Then

$$g \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)|$$

by Lemma 3.16, so that $g \leq 5 - |\text{Sing}(C)|$. This implies that C is smooth, because there are no \mathfrak{S}_5 -orbits of length less than 12 by Lemma 6.1.

Since \mathfrak{S}_5 does not act faithfully on \mathbb{P}^1 , we see that $g \neq 0$. Thus, either $g = 4$ or $g = 5$ by [6, Lemma 5.1.5]. The former case is impossible by Lemma 3.17, while the latter case is impossible by Lemma 4.4. \square

Recall from Section 3 that the subgroup $\mathfrak{A}_4 \subset \mathfrak{A}_5 \subset \mathfrak{S}_5$ fixes two disjoint lines L_1 and L'_1 . As before, we consider the \mathfrak{A}_5 -orbit L_1, \dots, L_5 of the line L_1 and the \mathfrak{A}_5 -orbit L'_1, \dots, L'_5 of the line L'_1 . By Lemma 3.7 the lines $L_1, \dots, L_5, L'_1, \dots, L'_5$ form a double five configuration (see Definition 3.6). Corollary 2.1(i) implies that the \mathfrak{S}_5 -orbit of the line L_1 is $L_1, \dots, L_5, L'_1, \dots, L'_5$.

Remark 6.3. Any subgroup $F_{20} \subset \mathfrak{S}_5$ permutes the ten lines $L_1, \dots, L_5, L'_1, \dots, L'_5$ transitively. Indeed, let $c \in F_{20}$ be an element of order five. Then c is not contained in a stabilizer of the line L_1 , so that the orbit of L_1 with respect to the group $\Gamma \cong \mu_5$ generated by c is L_1, \dots, L_5 . Similarly, the Γ -orbit of the line L'_1 is L'_1, \dots, L'_5 . Also, the group F_{20} is not contained in \mathfrak{A}_5 , so that the F_{20} -orbit of L_1 contains some of the lines L'_1, \dots, L'_5 , and thus contains all the ten lines $L_1, \dots, L_5, L'_1, \dots, L'_5$.

Let \mathcal{M} be the linear system on \mathbb{P}^3 consisting of all quartic surfaces passing through all lines L_1, \dots, L_5 and L'_1, \dots, L'_5 . Then \mathcal{M} is not empty. In fact, Lemma 3.7 and parameter count imply that its dimension is at least four. Moreover, the linear system \mathcal{M} does not have base components by Lemma 3.3.

Lemma 6.4. *The base locus of \mathcal{M} does not contain curves that are different from the lines $L_1, \dots, L_5, L'_1, \dots, L'_5$. Moreover, general surface in \mathcal{M} is smooth in a general point of each of these lines. Furthermore, two general surfaces in \mathcal{M} intersect transversally at a general point of each of these lines.*

Proof. Denote by Z the union of all curves that are contained in the base locus of the linear system \mathcal{M} and are different from the lines $L_1, \dots, L_5, L'_1, \dots, L'_5$. Then Z is a (possibly empty) \mathfrak{S}_5 -invariant curve. Denote its degree by d . Pick two general surfaces M_1 and M_2 in \mathcal{M} . Then

$$M_1 \cdot M_2 = Z + m \sum_{i=1}^5 L_i + m \sum_{i=1}^5 L'_i + \Delta,$$

where m is a positive integer, and Δ is an effective one-cycle on \mathbb{P}^3 that contains none of the lines L_1, \dots, L_5 and L'_1, \dots, L'_5 . Note that Δ may contain irreducible components of the curve Z . Note also that $\Delta \neq 0$, because \mathcal{M} is not a pencil.

Let Π be a plane in \mathbb{P}^3 . Then

$$16 = \Pi \cdot Z + m \sum_{i=1}^5 \Pi \cdot L_i + m \sum_{i=1}^5 \Pi \cdot L'_i + \Pi \cdot \Delta = d + 10m + \Pi \cdot \Delta > d + 10m,$$

which implies that $m = 1$ and $d \leq 5$. By Lemma 6.2, we have $d = 0$, so that Z is empty. Since

$$1 \geq m \geq \text{mult}_{L_i}(M_1 \cdot M_2) \geq \text{mult}_{L_i}(M_1) \text{mult}_{L_i}(M_2),$$

we see that a general surface in \mathcal{M} is smooth at a general point of L_i , and two general surfaces in \mathcal{M} intersect transversally at a general point of L_i . Similarly, we see that a general surface in \mathcal{M} is smooth at a general point of L'_i , and two general surfaces in \mathcal{M} intersect transversally at a general point of L'_i . \square

Let $g: W \rightarrow \mathbb{P}^3$ be a blow up along the lines L_1, \dots, L_5 , and let $g': W' \rightarrow \mathbb{P}^3$ be a blow up along the lines L'_1, \dots, L'_5 . Denote by $\tilde{L}_1, \dots, \tilde{L}_5$ (respectively, $\tilde{L}'_1, \dots, \tilde{L}'_5$) the proper transforms of the lines L_1, \dots, L_5 (respectively, on the threefold W'). Let $h: V \rightarrow W$ be a blow up along the curves $\tilde{L}_1, \dots, \tilde{L}_5$, and let $h': V' \rightarrow W'$ be a blow up along the curves $\tilde{L}'_1, \dots, \tilde{L}'_5$. Finally, let $\alpha: U \rightarrow \mathbb{P}^3$ be a blow up of the (singular) curve that is a union of all lines $L_1, \dots, L_5, L'_1, \dots, L'_5$. Then U has twenty nodes by Lemma 3.7, and there exists a commutative diagram

$$\begin{array}{ccccc} & V & \overset{\tau}{\dashrightarrow} & V' & \\ & \swarrow h & & \searrow h' & \\ W & & U & & W' \\ & \swarrow g & \downarrow \alpha & \searrow g' & \\ & \mathbb{P}^3 & & & \end{array}$$

(Note: The diagram shows arrows from V to W (h), V to U (v), V to V' (tau), V' to U (v'), V' to W' (h'), W to U (g), W to P^3 (g'), U to P^3 (alpha), and W' to P^3 (g').)

where v and v' are small resolutions of singularities of the threefold U , and τ is a composition of twenty Atiyah flops.

Remark 6.5. By construction, the action of group \mathfrak{A}_5 lifts to the threefolds W , W' , V , V' , and U . Similarly, the action of the group \mathfrak{S}_5 lifts to the threefold U , but this action does not lift to W and W' . On the threefolds V and V' , the group \mathfrak{S}_5 acts biregularly outside of the curves flopped by τ and τ^{-1} , respectively.

Denote by E_1, \dots, E_5 the g -exceptional surfaces that are mapped to L_1, \dots, L_5 , respectively. Similarly, denote by E'_1, \dots, E'_5 the g' -exceptional surfaces that are mapped to L'_1, \dots, L'_5 , respectively. Then all surfaces $E_1, \dots, E_5, E'_1, \dots, E'_5$ are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Denote by $\hat{E}'_1, \dots, \hat{E}'_5$ the h -exceptional surfaces that are mapped to the curves $\tilde{L}'_1, \dots, \tilde{L}'_5$, respectively. Similarly, denote by $\check{E}_1, \dots, \check{E}_5$ the h' -exceptional surfaces that are mapped to the curves $\tilde{L}_1, \dots, \tilde{L}_5$, respectively. Denote by $\hat{E}_1, \dots, \hat{E}_5$ the proper transforms on V of the surfaces E_1, \dots, E_5 , respectively. Finally, denote by $\check{E}'_1, \dots, \check{E}'_5$ the proper transforms on V' of the surfaces E'_1, \dots, E'_5 , respectively. Then τ maps the surfaces $\hat{E}_1, \dots, \hat{E}_5$ to the surfaces $\check{E}_1, \dots, \check{E}_5$, respectively, and it maps the surfaces $\hat{E}'_1, \dots, \hat{E}'_5$ to the surfaces $\check{E}'_1, \dots, \check{E}'_5$, respectively.

Denote by \mathcal{M}_W , \mathcal{M}_V , $\mathcal{M}_{W'}$, $\mathcal{M}_{V'}$, and \mathcal{M}_U the proper transforms of the linear system \mathcal{M} on the threefolds W , V , W' , V' , and U , respectively. Then it follows from Lemma 6.4 that

$$\mathcal{M}_W \sim -K_W, \quad \mathcal{M}_V \sim -K_V, \quad \mathcal{M}_{W'} \sim -K_{W'}, \quad \mathcal{M}_{V'} \sim -K_{V'},$$

and $\mathcal{M}_U \sim -K_U$.

Lemma 6.6. *The base locus of the linear system \mathcal{M}_W does not contain curves that are different from the curves $\tilde{L}'_1, \dots, \tilde{L}'_5$. Similarly, the base locus of $\mathcal{M}_{W'}$ does not contain curves that are different from the curves $\tilde{L}_1, \dots, \tilde{L}_5$.*

Proof. It is enough to prove the first assertion of the lemma. Suppose that the base locus of the linear system \mathcal{M}_W contains an irreducible curve Z that is different from the curves $\tilde{L}'_1, \dots, \tilde{L}'_5$. Then Z is contained in one of the surfaces E_1, \dots, E_5 by Lemma 6.4.

By Lemma 6.4, the curve Z is a fiber of some of the natural projections $E_i \rightarrow L_i$, because otherwise two general surfaces in \mathcal{M}_W would be tangent in a general point of L_i . In particular, the only curves in the base locus of the linear system \mathcal{M}_W are \tilde{L}'_i and possibly some fibers of the projections $E_i \rightarrow L_i$. This shows that $-K_W$ is nef. Indeed, $-K_W$ has positive intersections with the fibers of the projections $E_i \rightarrow L_i$, it has trivial intersection with all curves $\tilde{L}'_1, \dots, \tilde{L}'_5$, and $-K_W \sim \mathcal{M}_W$ has non-negative intersection with any other curve.

Let $Z_1 = Z, Z_2, \dots, Z_r$ be the \mathfrak{A}_5 -orbit of the curve Z . Then $r \geq 20$ by Corollary 3.5. Pick two general surfaces M_1 and M_2 in the linear system \mathcal{M}_W . By Lemma 6.4, one has

$$M_1 \cdot M_2 = \sum_{i=1}^5 \tilde{L}'_i + m \sum_{i=1}^r Z_i + \Delta$$

for some positive integer m and some effective one-cycle Δ whose support contains none of the curves $\tilde{L}'_1, \dots, \tilde{L}'_5$ and Z_1, \dots, Z_r . Hence

$$\begin{aligned} 14 = -K_W^3 &= -K_W \cdot M_1 \cdot M_2 = -K_W \cdot \left(\sum_{i=1}^5 \tilde{L}'_i + m \sum_{i=1}^r Z_i + \Delta \right) = \\ &= -5K_W \cdot \tilde{L}'_1 - mrK_W \cdot Z - K_W \cdot \Delta = mr - K_W \cdot \Delta \geq mr \geq r \geq 20, \end{aligned}$$

which is absurd. \square

Lemma 6.7. *The linear system \mathcal{M}_V is base point free.*

Proof. It is enough to show that \mathcal{M}_V is free from base curves. Indeed, if the base locus of the linear system \mathcal{M}_V does not contain base curves, then \mathcal{M}_V cannot have more than $-K_V^3 = 4$ base points, because $\mathcal{M}_V \sim -K_V$. On the other hand, V does not contain \mathfrak{S}_5 -orbits of length less than 12, because there are no \mathfrak{S}_5 -orbits of such length on \mathbb{P}^3 by Lemma 6.1.

Suppose that the base locus of the linear system \mathcal{M}_V contains an irreducible curve Z . If Z is not contained in any of the surfaces $\hat{E}'_1, \dots, \hat{E}'_5$, then the curve $h(Z)$ is a base curve of the linear system \mathcal{M}_W and $h(Z)$ is different from the curves $\tilde{L}'_1, \dots, \tilde{L}'_5$. This is impossible by Lemma 6.6. Similarly, if Z is not contained in any of the surfaces $\hat{E}_1, \dots, \hat{E}_5$, then the curve $h' \circ \tau(Z)$ is a base curve of the linear system $\mathcal{M}_{W'}$ that is different from the curves $\tilde{L}_1, \dots, \tilde{L}_5$. This is again impossible by Lemma 6.6. Thus, Z is contained in one of the surfaces $\hat{E}_1, \dots, \hat{E}_5$, and in one of the surfaces $\hat{E}'_1, \dots, \hat{E}'_5$. In particular, the curves flopped by τ are not contained in the base locus of \mathcal{M}_V .

Without loss of generality, we may assume that $Z = \hat{E}_1 \cap \hat{E}'_2$. Let C be the curve flopped by τ that is contained in \hat{E}_1 and intersects \hat{E}'_2 . Then C intersects Z by one point. On the other hand, we have $-K_V \cdot C = 0$. Since $\mathcal{M}_V \sim -K_V$, this implies that C is disjoint from a general surface in \mathcal{M}_V . This is impossible, because $C \cap Z \neq \emptyset$, while Z is contained in the base locus of the linear system \mathcal{M}_V . \square

Corollary 6.8. *The linear systems $\mathcal{M}_{V'}$, and \mathcal{M}_U are also base point free.*

Proof. Recall that $\mathcal{M}_V \sim -K_V$. Thus, the general surface of \mathcal{M}_V is disjoint from all curves flopped by τ , because \mathcal{M}_V is base point free by Lemma 6.7. \square

Corollary 6.9. *The base locus of \mathcal{M} consists of the lines $L_1, \dots, L_5, L'_1, \dots, L'_5$.*

By Lemma 6.7 and Corollary 6.8, the divisors $-K_V$, $-K_{V'}$, and $-K_U$ are nef. Since

$$-K_V^3 = -K_{V'}^3 = -K_U^3 = 4,$$

these divisors are also big. Thus, the Kawamata–Viehweg vanishing theorem and the Riemann–Roch formula give

$$(6.10) \quad h^0(\mathcal{O}_V(-K_V)) = h^0(\mathcal{O}_{V'}(-K_{V'})) = h^0(\mathcal{O}_U(-K_U)) = 4.$$

In particular, one has $|-K_V| = \mathcal{M}_V$, $|-K_{V'}| = \mathcal{M}_{V'}$, and $|-K_U| = \mathcal{M}_U$.

Lemma 6.11. *The \mathfrak{S}_5 -representation $H^0(\mathcal{O}_U(-K_U))$ is irreducible.*

Proof. By Lemma 3.14, there are no \mathfrak{S}_5 -invariant quartic surfaces in \mathbb{P}^3 that pass through the ten lines $L_1, \dots, L_5, L'_1, \dots, L'_5$. This implies that $H^0(\mathcal{O}_U(-K_U))$ does not contain one-dimensional subrepresentations. Hence it is irreducible by Remark 4.5. \square

Lemma 6.7 together with (6.10) implies that there is an \mathfrak{S}_5 -equivariant commutative diagram

$$(6.12) \quad \begin{array}{ccc} & U & \\ \alpha \swarrow & & \searrow \beta \\ \mathbb{P}^3 & \dashrightarrow \phi & \mathbb{P}^4, \end{array}$$

where ϕ is a rational map given by \mathcal{M} , and β is a morphism given by the anticanonical linear system $|-K_U|$. By Lemma 6.11 the projective space \mathbb{P}^4 in (6.12) is a projectivization of an irreducible \mathfrak{S}_5 -representation.

For $1 \leq i < j \leq 5$, let Λ_{ij} be the intersection line of the plane spanned by L_i and L'_j with the plane spanned by L'_i and L_j . Note that the stabilizer of Λ_{ij} in \mathfrak{S}_5 contains a subgroup isomorphic to D_{12} . Actually, this implies that the stabilizer of Λ_{ij} in \mathfrak{S}_5 is isomorphic to D_{12} , since D_{12} is a maximal proper subgroup in \mathfrak{S}_5 (see Remark 4.3) and there are no \mathfrak{S}_5 -invariant lines in \mathbb{P}^3 by Corollary 2.1(i). Denote by $\hat{\Lambda}_{ij}$ the proper transform of the line Λ_{ij} on the threefold V , and denote by $\bar{\Lambda}_{ij}$ its proper transform on U . Then

$$-K_V \cdot \hat{\Lambda}_{ij} = 0.$$

Since v is a small birational morphism, we also obtain $-K_U \cdot \bar{\Lambda}_{ij} = 0$.

We see that each curve $\bar{\Lambda}_{ij}$ is contracted by β to a point. Note that the stabilizer of Λ_{ij} in \mathfrak{S}_5 is isomorphic to D_{12} . Since $-K_U^3 = 4$, the image of β is either an \mathfrak{S}_5 -invariant quartic threefold or an \mathfrak{S}_5 -invariant quadric threefold. Applying Corollary 4.8 together with Lemma 4.11, we obtain the following.

Corollary 6.13. *The morphism β is birational on its image, and its image is a quartic threefold.*

Now Lemmas 4.11 and 4.12 imply that the image of β is the quartic $X_{\frac{1}{6}}$. This proves

Corollary 6.14. *The threefold $X_{\frac{1}{6}}$ is rational.*

Remark 6.15. An alternative approach to the rationality of the quartic threefold $X_{\frac{1}{6}}$ was suggested in [16]. Unfortunately, its implementation seems to contradict the existence of the commutative diagram (6.12). Indeed, the paper [16] studies the action of the subgroup $F_{20} \subset \mathfrak{S}_6$ on the threefold $X_{\frac{1}{6}}$. Since all such subgroups in \mathfrak{S}_6 are conjugate, one may identify F_{20} with a subgroup of our \mathfrak{S}_5 . By Remark 6.3, the group F_{20} permutes the ten lines $L_1, \dots, L_5, L'_1, \dots, L'_5$ transitively. This means that γ is an $F_{20}\mathbb{Q}$ -factorialization of the quartic threefold $X_{\frac{1}{6}}$. Thus, the application of F_{20} -Minimal Model Program to U must give the birational map $\alpha: U \rightarrow \mathbb{P}^3$. However, [16, Lemma 2.10], [16, Lemma 2.12] and [16, Lemma 2.13] exclude this possibility.

Ten curves $\bar{\Lambda}_{ij}$ are mapped by γ to ten singular points of the threefold $X_{\frac{1}{6}}$. Twenty singular points of U are mapped by γ to another twenty singular points of $X_{\frac{1}{6}}$. Let us describe the curves in U that are contracted by γ to the remaining ten singular points of the threefold $X_{\frac{1}{6}}$. To do this, we need

Lemma 6.16. *Let ℓ_1, ℓ_2, ℓ_3 and ℓ_4 be pairwise skew lines in \mathbb{P}^3 . Suppose that there is a unique line $\ell \subset \mathbb{P}^3$ that intersects ℓ_1, ℓ_2, ℓ_3 and ℓ_4 . Let $\pi: Y \rightarrow \mathbb{P}^3$ be a blow up of the*

line ℓ , and $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ be the exceptional divisor of π . Denote by $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3$ and $\tilde{\ell}_4$ the proper transforms on Y of the lines ℓ_1, ℓ_2, ℓ_3 and ℓ_4 , respectively. Then there exists a unique curve $C \subset E$ of bi-degree $(1, 1)$ that intersects the curves $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3$ and $\tilde{\ell}_4$.

Proof. The lines ℓ_1, ℓ_2 , and ℓ_3 are contained in a unique quadric surface $S \subset \mathbb{P}^3$. Note that S is smooth, because ℓ_1, ℓ_2 , and ℓ_3 are disjoint. Furthermore, the line ℓ is contained in S , because ℓ intersects the lines ℓ_1, ℓ_2 , and ℓ_3 by assumption. Moreover, the line ℓ_4 is tangent to S , since otherwise there would be either two or infinitely many lines in \mathbb{P}^3 that intersect ℓ_1, ℓ_2, ℓ_3 and ℓ_4 . Denote by \tilde{S} the proper transform on Y of the quadric surface S . Then \tilde{S} contains the curves $\tilde{\ell}_1, \tilde{\ell}_2$, and $\tilde{\ell}_3$. Moreover, \tilde{S} intersects the curve $\tilde{\ell}_4$. Thus $\tilde{S}|_E$ is the required curve C . \square

By Lemmas 3.7 and 6.16, each surface $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ contains a unique smooth rational curve C_i of bi-degree $(1, 1)$ that passes through all four points of the intersection of E_i with the curves $\tilde{L}'_1, \dots, \tilde{L}'_5$ (recall that $E_i \cap \tilde{L}'_i = \emptyset$). Similarly, each surface $E'_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ contains a unique smooth rational curve C'_i of bi-degree $(1, 1)$ that passes through all four points of the intersection of E'_i with the curves $\tilde{L}_1, \dots, \tilde{L}_5$. Denote by $\hat{C}_1, \dots, \hat{C}_5$ the proper transforms on the threefold V of the curves C_1, \dots, C_5 , respectively. Similarly, denote by $\check{C}'_1, \dots, \check{C}'_5$ the proper transforms on the threefold V' of the curves C'_1, \dots, C'_5 , respectively. Then

$$-K_V \cdot \hat{C}_i = -K_{V'} \cdot \check{C}'_i = 0.$$

This implies that the proper transforms of the curves $\hat{C}_1, \dots, \hat{C}_5$ on the threefold V' are (-2) -curves on the surfaces $\tilde{E}_1, \dots, \tilde{E}_5$, respectively. Similarly, the proper transforms of the curves $\check{C}'_1, \dots, \check{C}'_5$ on the threefold V are (-2) -curves on the surfaces $\hat{E}'_1, \dots, \hat{E}'_5$, respectively. Thus, all surfaces $\hat{E}'_1, \dots, \hat{E}'_5, \tilde{E}_1, \dots, \tilde{E}_5$ are isomorphic to the Hirzebruch surface \mathbb{F}_2 .

Denote by $\overline{C}_1, \dots, \overline{C}_5, \overline{C}'_1, \dots, \overline{C}'_5$ the images of the curves $\hat{C}_1, \dots, \hat{C}_5, \check{C}'_1, \dots, \check{C}'_5$ on the threefold U , respectively. Then

$$-K_U \cdot \overline{C}_i = -K_U \cdot \overline{C}'_i = 0,$$

because $-K_V \cdot \hat{C}_i = -K_{V'} \cdot \check{C}'_i = 0$, and v and v' are small birational morphisms. Thus, the ten curves $\overline{C}_1, \dots, \overline{C}_5, \overline{C}'_1, \dots, \overline{C}'_5$ are contracted by the morphism β to ten singular points of $X_{\frac{1}{6}}$.

It would be interesting to extend the commutative diagram (6.12) to an \mathfrak{S}_5 -Sarkisov link similar to the \mathfrak{A}_6 -Sarkisov link (5.20).

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